Extending Process Algebra with an undefined action

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July 8, 2022

Outline

- Motivation
- Process Model
- Basic Extended Process Algera
- Prebisimilarity
- Logical Characterisation: PHML
- Conclusion

Divergence, Livelock and Deadlock

- $\approx$ [8] and weaker equivalences [4] are insensitive to "τ-cycles".
- "τ-cycles" ($T \Leftarrow \tau.T$) are identified with divergence [11] and has the same solution as $X \Leftarrow X$.
- "τ-cycles" could also be due to "livelock" i.e. infinite "internal chatter". $P \Leftarrow a.P$, $Q \Leftarrow \overline{a}.Q$, $R = (P|Q)\backslash a \sim \tau.R$
- In SCCS [7] the solution of $X \Leftarrow X$ is identified with deadlock.

Theoretical Computer Science 25 (1983) 267–310 267
North-Holland

CALCULI FOR SYNCHRONY AND ASYNCHRONY*

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Communicated by M. Nivat
Received February 1982
Revised August 1982

The derivation rule for recursion is as follows:

$$\frac{E_i\{\mathrm{fix}\,\tilde{X}\tilde{E}/\tilde{X}\} \xrightarrow{a} E'}{\mathrm{fix}_i\,\tilde{X}\tilde{E} \xrightarrow{a} E'}$$

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(i) $F \equiv \mathrm{fix}\,XX$. The only instances of the rule which can yield an action for F have the form

$$\frac{F \xrightarrow{a} E'}{F \xrightarrow{a} E'}$$

Hence, since every derivation must be inferred by a finite proof, F in this case has no actions. It will turn out to be congruent to **0**.

Our view

Divergence (Ω). The least solution of $X \Leftarrow X$ (modulo $\sim$) should be a totally "undefined" process.

Livelock. The least solution (modulo $\sim$) of $T \Leftarrow \tau.T$ should be a process that can only perform "τ-cycles".

Deadlock ($\mathbf{0}$). A deadlocked process performs no computation unlike a live-locked process which consumes computational cycles and energy.

Hence any strong behavioural relation on processes should ensure that

- divergence, deadlock and livelock are distinguished from each other.
- Ω is the least defined process (modulo $\sim$),
- Deadlock ($\mathbf{0}$) and livelock ($T \Leftarrow \tau.T$) are both well-defined processes and mutually incomparable.

LTS

- $\mathbb{L}[L] = \langle S, L, \longrightarrow \rangle$
- $s \xrightarrow{\ell} \_ = \{t \in S \mid s \xrightarrow{\ell} t\}$ is the set of ℓ-*successors* of s.
- $L(s) = \{\ell \in L \mid \exists t[s \xrightarrow{\ell} t]\}$ is the set of labels from s.
- $Succ(s) = \bigcup_{\ell \in L} s \xrightarrow{\ell} \_ = \{t \mid \exists \ell \in L[s \xrightarrow{\ell} t]\}$ is the set of *successors* of s
- $Targets(\longrightarrow) = \{t \in S \mid \exists s \in S[t \in Succ(s)]\}$.
- $Der(s) = \{s\} \cup \bigcup_{t \in Succ(s)} Der(t)$ is the set of *derivatives* of s.

A **sub-LTS** of $\mathbb{L}[L]$ at a state $s_0 \in S$ is the rooted LTS $\langle Der(s_0), L, \longrightarrow, s_0 \rangle$.
By convention $s \xrightarrow{\epsilon} s$ and for any $x = ay \in L^+$, $s \xrightarrow{x} s'$ if $s \xrightarrow{a} s'' \xrightarrow{y} s'$ for some $s'' \in S$.

Natural Bisimilarity

Definition 0.1: Natural bisimulation

A symmetric binary relation $\mathcal{R} \subseteq S \times T$ between (sub-)LTSs $\mathbb{L}[L] = \langle S, L, \longrightarrow \rangle$ and $\mathbb{M}[L] = \langle T, L, \longrightarrow \rangle$ such that $s\mathcal{R}t$ implies for all labels $\ell \in L$, $s \stackrel{\ell}{\longrightarrow} s' \Rightarrow \exists t' \in T[t \stackrel{\ell}{\longrightarrow} t' \wedge s'\mathcal{R}t']$. Notation: $\mathcal{R} \vdash s \sim t$).

Fact 0.1

Unions, relational converses and (relational) compositions of natural bisimulations are also natural bisimulations. Natural bisimilarity ($\sim$) is the largest natural bisimulation and is an equivalence relation.

Actions and Traces

Definition 0.2: Actions

$A_\perp = A \cup \{\perp\}$ where A is a countable set of (uninterpreted but) *well-defined* actions and $\perp \notin A$ is a special *undefined action* with $\perp < a$ for each $a \in A$.

- Ω can perform only $\perp$.
- *Traces* are words from $A^* \perp^*$.
- $A^* \perp^? := A^* \perp^* / (x\perp\perp = x\perp)$ the set of normal forms of traces.

Definition 0.3: Ordering

$\leq \subseteq A^*\perp^? \times A^*\perp^?$ be the smallest relation such that for all $x, y \in A^*$, $x \leq x$ and $x\perp \leq xy\perp \leq xy$. $u < v$ if $u \leq v \not\leq u$ for all $u, v \in A^*\perp^?$.

Definition 0.4: Process

- A **(partial) process** is a sub-LTS $\langle Der(s_0), A_\perp, \longrightarrow, s_0\rangle$ satisfying the **irrecoverability** constraint

$$\forall s \in Der(s_0)[s \stackrel{\perp}{\longrightarrow} s' \Rightarrow A_\perp(s') = \{\perp\}] \qquad (1)$$

- The process is **total** if $s \stackrel{\perp}{\not\longrightarrow} s'$ for all $s, s' \in Der(s_0)$.
- If $s_0 \stackrel{u}{\longrightarrow} t$ for $t \in Der(s_0)$ and $u \in A^{*}\perp^{?}$, then $s_0 \stackrel{u}{\longrightarrow} t$ is a **behaviour** of the process.

Fact 0.2: Closed-under-transitions

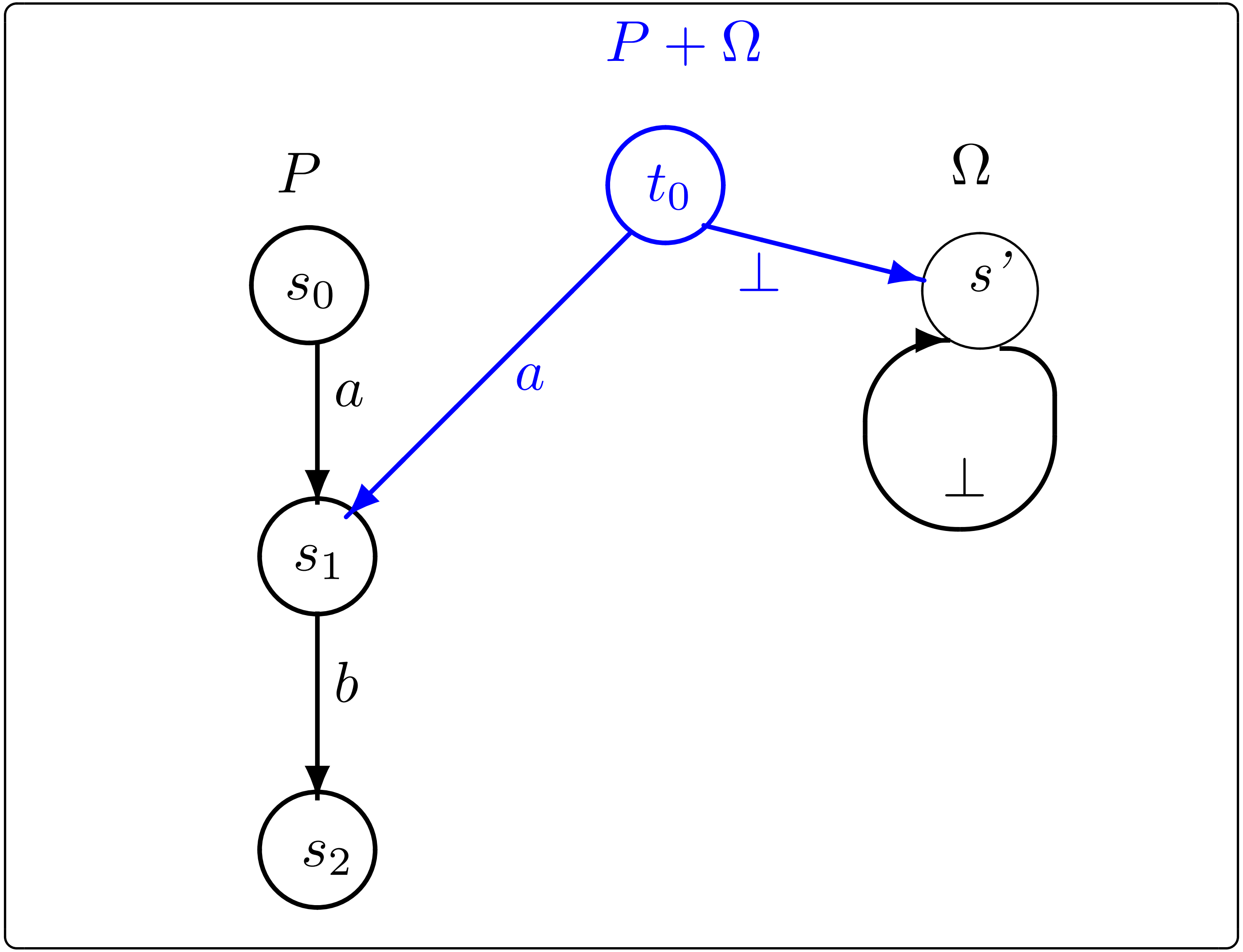
If $\langle S, A_\perp, \longrightarrow, s_0\rangle$ is a process then so is $\langle Der(s), A_\perp, \longrightarrow, s\rangle$ for any $s \in S$.

Definition 0.5: Basic Extended Process Algebra (BXPA)

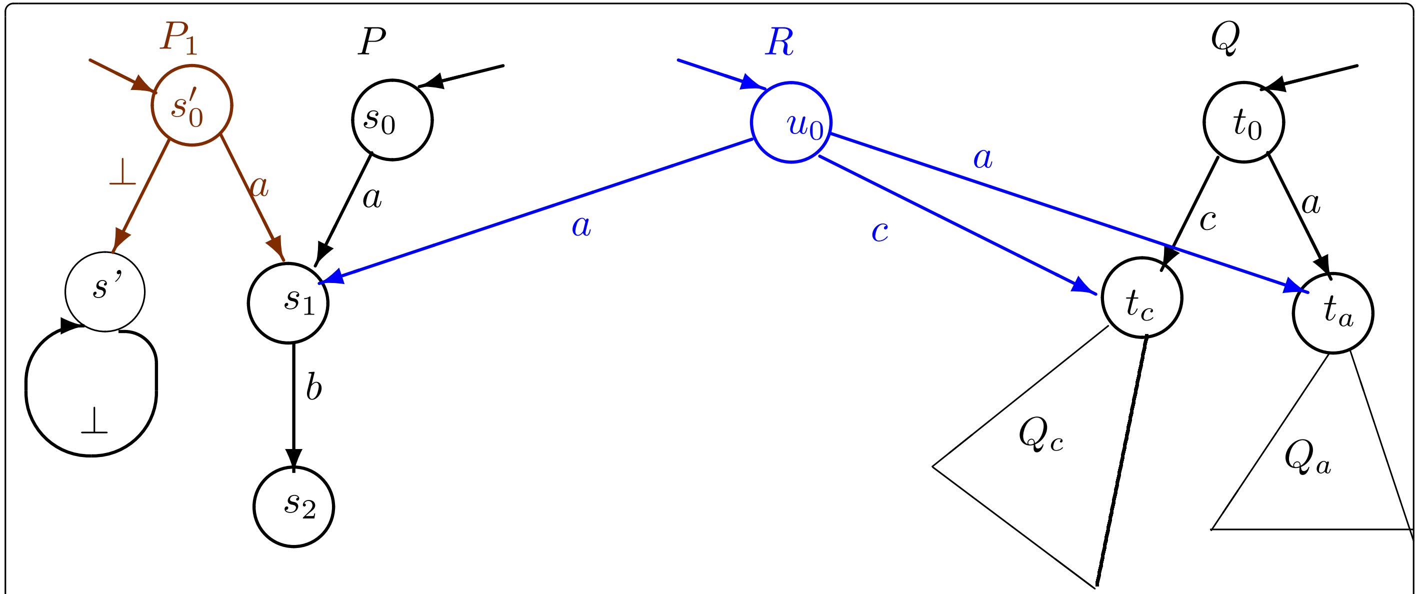
$\mathbf{P}[A_\perp] = \langle \mathbb{P}[A_\perp], \Omega, \mathbf{0}, \{a.\_ \mid a \in A\}, \sum \rangle$ where

- $\Omega \stackrel{df}{=} \langle \{s_0\}, A_\perp, \{s_0 \stackrel{\perp}{\longrightarrow} s_0\}, s_0 \rangle$ the totally undefined process,
- $\mathbf{0} \stackrel{df}{=} \langle \{s_0\}, A_\perp, \emptyset, s_0 \rangle$ is the "terminated" or "deadlocked" process.
- $a.P \stackrel{df}{=} \langle S \cup \{s'_0\}, A_\perp, \longrightarrow \cup \{s'_0 \stackrel{a}{\longrightarrow} s_0\}, s'_0 \rangle$, for any $P = \langle S, A_\perp, \longrightarrow, s_0 \rangle$, $a \in A$, and $s'_0 \notin S$,
- $\sum_{i \in I} P_i \stackrel{df}{=} \langle S, A_\perp, \longrightarrow s_0 \rangle$ where $P_i = \langle S^i, A_\perp, \longrightarrow_i, s^i_0 \rangle$, $i \in I$ and
  - $s_0 \notin \bigcup_{i \in I} S^i$ and $S = Der(s_0) = \{s_0\} \cup \biguplus_{i \in I} Targets(\longrightarrow_i)$,
  - $s_0 \stackrel{a}{\longrightarrow} t$ if for some P_i, $i \in I$, $s^i_0 \stackrel{a}{\longrightarrow}_i t \in S^i$,
  - $s \stackrel{a}{\longrightarrow} t$ if $s \stackrel{a}{\longrightarrow}_i t$ for some $i \in I$, $s, t \in Der(s_0)$.

Example: Summation 1



Example: Summation 2



$P \equiv a.b.\mathbf{0}$

$P_1 \equiv \Omega + P$

$Q \equiv c.Q_c + a.Q_a$

$R \equiv P + Q$

BXPA: Basic Identities

Proposition 0.1

$\mathbb{P}[A_\perp]$ is an idempotent abelian monoid under $+$ with $\mathbf{0}$ as identity. Further

1. $P \xrightarrow{a} P', a \in A$ implies $P \sim a.P' + P$.
2. $P \xrightarrow{\perp} P'$ implies $P' \sim \Omega$ and hence $P \sim \Omega + P$.
3. **(Canonical form modulo $\sim$).** $P \sim [\Omega+] \sum_{a \in A, P \xrightarrow{a} P_a} a.P_a$

   where "$[\Omega+]$" indicates that Ω occurs only if $P \xrightarrow{\perp}$.

Composition

Strictness condition (see irrecoverability)

$$(P \xrightarrow{\perp} \_ \vee Q \xrightarrow{\perp} \_) \implies ((P \otimes Q \xrightarrow{\perp} \Omega) \wedge (Q \otimes P \xrightarrow{\perp} \Omega)) \qquad (2)$$

guarantees that $\mathbb{P}[A_\perp]$ is closed under $\otimes$. Hence expansion laws under the various composition operators (e.g. $|||$, $|$, $\|$, $\times$) continue to hold.

Lifted Strong Bisimulation (LSB)

Definition 0.6: lifted strong bisimulations (LSB)

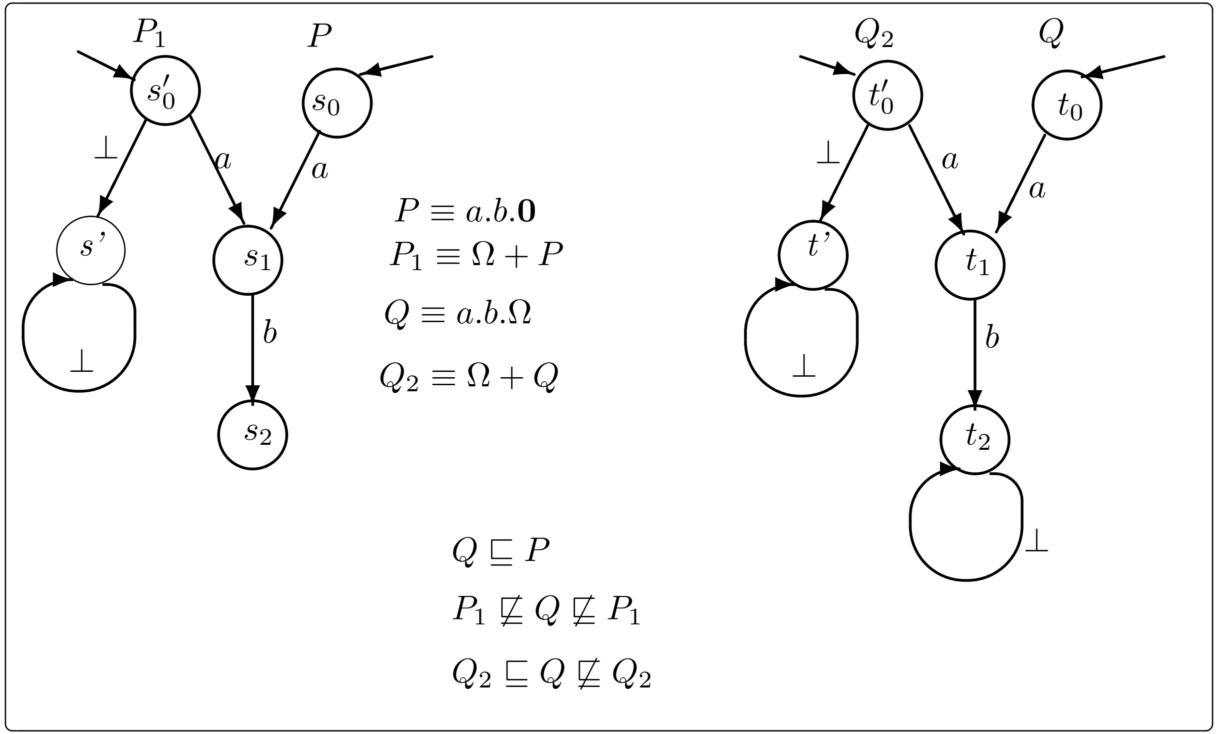
A binary relation $\mathcal{R}$ on processes is a **lifted strong bisimulation (LSB)** if for all states s, t, $s\mathcal{R}t$ implies for all $a, b \in A_{\perp,\epsilon}$,

$$s \xrightarrow{a} s' \Rightarrow \exists b, t'[a \leq b \wedge t \xrightarrow{b} t' \wedge s'\mathcal{R}t'] \quad (3)$$

$$t \xrightarrow{b} t' \Rightarrow \exists a, s'[a \leq b \wedge s \xrightarrow{a} s' \wedge s'\mathcal{R}t'] \quad (4)$$

- $s \sqsubseteq t$ (equivalently $t \sqsupseteq s$) if there exists a LSB $\mathcal{R}$ such that $s\mathcal{R}t$.
- $s \sqsubseteq\!\!\!\sqsupseteq t$ if $s \sqsubseteq t$ and $s \sqsupseteq t$.
- $s \sqsubset t$ if $s \sqsubseteq t$ and $s \not\sqsupseteq t$.

Examples



In addition, if τ is in A and $\mathbf{T} \Leftarrow \tau.\mathbf{T}$ denotes livelock, then we have $\bot < \tau$, $\bot < \epsilon$ and hence $\Omega \sqsubset \mathbf{0} \not\sqsubseteq \Omega$, $\Omega \sqsubset \mathbf{T} \not\sqsubseteq \Omega$ and $\mathbf{0} \not\sqsubseteq \mathbf{T} \not\sqsubseteq \mathbf{0}$

Precongruence

LSB is an instance of the more general (ρ, σ)-bisimulation [1] with $\rho = \sigma = \leq$. By theorem 4.1 part 1 in [1], $\sqsubseteq$ is a preorder.

Theorem 0.1: Precongruence.

The operators of $\mathbf{P}[A_\perp]$ are monotonic under $\sqsubseteq$ and the relation $\sqsubseteq$ is a precongruence on $\mathbf{P}[A_\perp]$.

Logical Characterisation

Definition 0.7

- $\mathcal{L}$ a logical language and
- $\models^X \subseteq \mathbb{P} \times \mathcal{L}$ a satisfaction relation
- $\mathcal{L}_X(P) = \{\phi \in \mathcal{L} \mid P \models^X \phi\}$
- $(\mathcal{L}, \models^X)$ **characterises** a behavioural preorder $\preceq$ over $\mathbb{P}$

$$P \preceq Q \Leftrightarrow \mathcal{L}_X(P) \subseteq \mathcal{L}_X(Q) \tag{5}$$

- $P \subseteq_X Q$ iff $\mathcal{L}_X(P) \subseteq \mathcal{L}_X(Q)$.

PHML: A Modal logic

Definition 0.8

- Negation-free modal logic $\mathcal{L}_{(\leq,\leq)}$[a]

$$\phi \;::=\; \mathtt{tt} \mid \mathtt{ff} \mid \langle a\rangle\phi \mid [a]\phi \mid \bigwedge_{i\in I}\phi_i \mid \bigvee_{i\in I}\phi_i \tag{6}$$

where $a \in A_{\perp,\epsilon}$ and I is an indexing set,

- $\bigwedge_{i\in\emptyset}\phi_i \equiv \mathtt{tt}$ and $\bigvee_{i\in\emptyset}\phi_i \equiv \mathtt{ff}$ by convention.

[a] For the present, we are assuming that every action in $A_{\perp,\epsilon}$ including the undefined action $\perp$ is observable; this may be relaxed.

Semantics: Satisfaction

Definition 0.9: Satisfaction

$P \models^S \mathit{tt}$ for each $P \in \mathbb{P}_{IF}$

$P \models^S \langle a \rangle \phi$ iff

$\exists b \in A_{\perp,\epsilon} : b \geq a, P' :$

$[P \xrightarrow{b} P' \wedge P' \models^S \phi]$

$P \models^S \bigwedge_{i \in I} \phi_i$ iff $\forall i \in I[P \models^S \phi_i]$

$P \models^S \mathit{ff}$ for no $P \in \mathbb{P}_{IF}$

$P \models^S [a]\phi$ iff

$\forall b \in A_{\perp,\epsilon} : b \leq a, P' :$

$[P \xrightarrow{b} P' \Rightarrow P' \models^S \phi]$

$P \models^S \bigvee_{i \in I} \phi_i$ iff $\exists i \in I[P \models^S \phi_i]$

- P **satisfies** ϕ if $P \models^S \phi$ and
- $\mathcal{L}_S(P) = \{\phi \mid P \models^S \phi\}$
- $P \subseteq_S Q$ if $\mathcal{L}_S(P) \subseteq \mathcal{L}_S(Q)$ for processes P, Q,

PHML characterisation of LSB

Theorem 0.2: Logical characterisation of $\sqsubseteq$

$P \sqsubseteq Q$ if and only if $\mathcal{L}_S(P) \subseteq \mathcal{L}_S(Q)$ for image-finite processes.

Theorem 0.2 then directly follows from definition 5 and theorem 3 of [3].

But

- it is difficult to explain modalities like $\langle\bot\rangle$ and $[\bot]$.
- Also modalities like $\langle\epsilon\rangle$ and $[\epsilon]$ do not add much value to the notion of observation.

Redefining old notions

- s may **diverge** ($s\uparrow$) if $s \xrightarrow{\perp}$. Otherwise it converges ($s\downarrow$). Analogously for processes.
- A binary relation $\mathcal{R}$ on processes is a **divergent strong bisimulation (DSB)** if for all $s, t \in S$, $s\mathcal{R}t$ implies the following.

$$\forall a \in A[s \xrightarrow{a} s' \Rightarrow \exists t'[t \xrightarrow{a} t' \wedge s'\mathcal{R}t'] \tag{7}$$

$$s\downarrow \Rightarrow (t\downarrow \wedge \forall a \in A[t \xrightarrow{a} t' \Rightarrow \exists s'[s \xrightarrow{a} s' \wedge s'\mathcal{R}t']]) \tag{8}$$

- $s \sqsubseteq\!\!\sim t$ (equivalently $t \sqsupseteq\!\!\sim s$) if there exists a DSB $\mathcal{R}$ such that $s\mathcal{R}t$ (we write $\mathcal{R} \vdash s \sqsubseteq\!\!\sim t$ to denote this fact). $s \sqsubset\!\!\sqsupset\!\!\sim t$ if $s \sqsubseteq\!\!\sim t$ and $s \sqsupseteq\!\!\sim t$.

Equivalence of LSB and DSB

- $\{\Omega\} \times \mathbb{P}$ is a DSB and hence $\Omega \underset{\sim}{\sqsubset} P$ for all $P \in \mathbb{P}$.
- ($\mathcal{R}$ completion) $\mathcal{R}^{\perp} = \mathcal{R} \cup \{(s, t') \mid s\mathcal{R}t, s\uparrow, A(s) = \emptyset, t\downarrow, t' \in Der(t)\}$.

Lemma 0.1

1. If $\mathcal{R}$ is a DSB then so is $\mathcal{R}^{\perp}$.
2. $\mathcal{R}$ is a DSB implies $\mathcal{R}^{\perp}$ is a LSB.
3. If $\mathcal{R}$ is a LSB then so is $\mathcal{R}^{\perp}$.
4. Every LSB is also a DSB.

Theorem 0.3

$\underset{\sim}{\sqsubset} = \sqsubseteq$ and $\underset{\sim}{\sqcup} = \underline{\sqcup}$.

$\mathcal{L}^{-\perp}$ is $\mathcal{L}$ without $\langle\perp\rangle$ and $[\perp]$.

Definition 0.10: Affirmation

$\models^A \subseteq \mathbb{P} \times \mathcal{L}^{-\perp}$ is the smallest (infix) relation defined by induction on the structure of formulae for any process P and any action $a \in A_{\perp,\epsilon}$.

$P \models^A \mathtt{tt}$ for each $P \in \mathbb{P}$

$P \models^A \mathtt{ff}$ for no $P \in \mathbb{P}$

$P \models^A \langle a\rangle\phi$ iff $\exists P'[P \xrightarrow{a} P' \wedge P' \models^A \phi]$

$P \models^A [a]\phi$ iff $P\!\downarrow \wedge\, \forall P'[P \xrightarrow{a} P' \Rightarrow P' \models^A \phi]$

$P \models^A \bigwedge_{i\in I} \phi_i$ iff $\forall i \in I[P \models^A \phi_i]$

$P \models^A \bigvee_{i\in I} \phi_i$ iff $\exists i \in I[P \models^A \phi_i]$

P **affirms** ϕ if $P \models^A \phi$ and $\mathcal{L}_A(P) = \{\phi \mid P \models^A \phi\}$. $P \subseteq_A Q$ if $\mathcal{L}_A(P) \subseteq \mathcal{L}_A(Q)$ for processes P and Q.

Characterisation

Definition 0.11

1. $P \subseteq_S^{-\perp} Q$ iff $\mathcal{L}_S^{-\perp}(P) \subseteq \mathcal{L}_S^{-\perp}(Q)$.
2. $P \subseteq_A^{-\perp} Q$ iff $\mathcal{L}_A^{-\perp}(P) \subseteq \mathcal{L}_A^{-\perp}(Q)$.

Theorem 0.4

$\boxed{\subseteq_S^{-\perp} = \sqsubseteq = \subseteq_A^{-\perp}}$ i.e. $\mathcal{L}^{-\perp}$ characterises the preorder $\sqsubseteq$.

Conclusions.

- Recursion not explicitly considered (since the model allows processes with infinite behaviours).
- But easy to see that guarded recursion (made up only of well-defined actions) will yield unique fixpoints.
- If $\tau \in A$ then $X \Leftarrow X$ and $X \Leftarrow \tau.X$ will yield different least solutions.

Future Work.

- $\sqsubset$ could be used as a refinement relation that allows the progression from a totally undefined process to a well-defined process satisfying certain modal properties.
- $\mathbb{P}[A_\perp]$ is closed under various composition operations. This allows the possibility of using more than one parallel composition operator in the specification of systems.

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Thank You!

Any Questions?