# A Logical Characterization of Efficiency Preorders

Neelesh Korade
neelesh_korade@persistent.co.in
Persistent Systems Private Limited
Pune, India 411 016.
and
S. Arun-Kumar
sak@cse.iitd.ernet.in
Department of Computer Science and Engineering
I. I. T. Delhi, Hauz Khas, New Delhi 110 016.
http://www.cse.iitd.ernet.in/∼sak

September 23, 2004

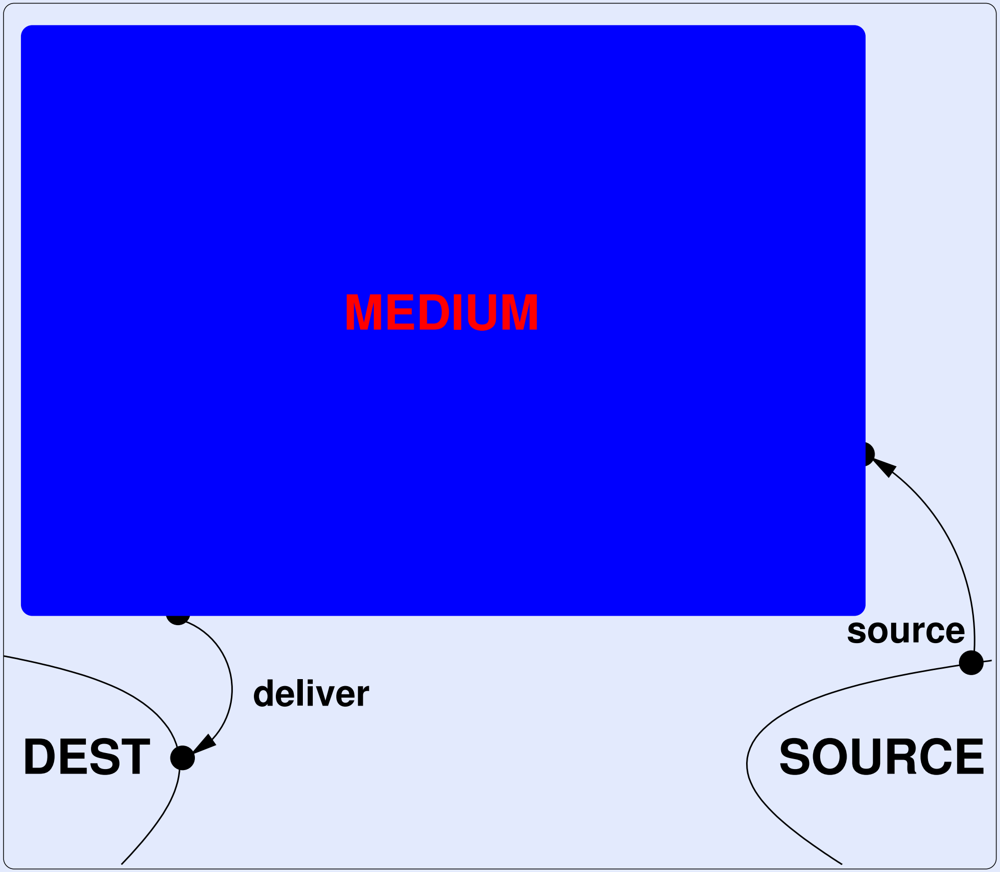
# Outline

- Introduction
- Efficiency-based Preorders
- Simulations & $HML^{-}$
- Principle of Containment
- Strong Bisimulations and $HML$
- Derived Operators & Expressiveness
- Weak Bisimulations & $OHML$
- Elaboration
- Efficiency Prebisimulations
- Conclusion

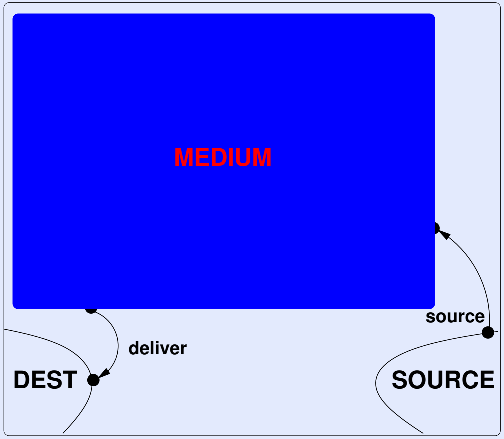
# Colour Coding

- Behavioural relations.
- Logic related symbols and terms.
- Hyperlinks or references
- Headings of slides and general highlighting
- anything that needs emphasis
- Terms being Defined
- Everything else.

# Example: 3buffer Medium

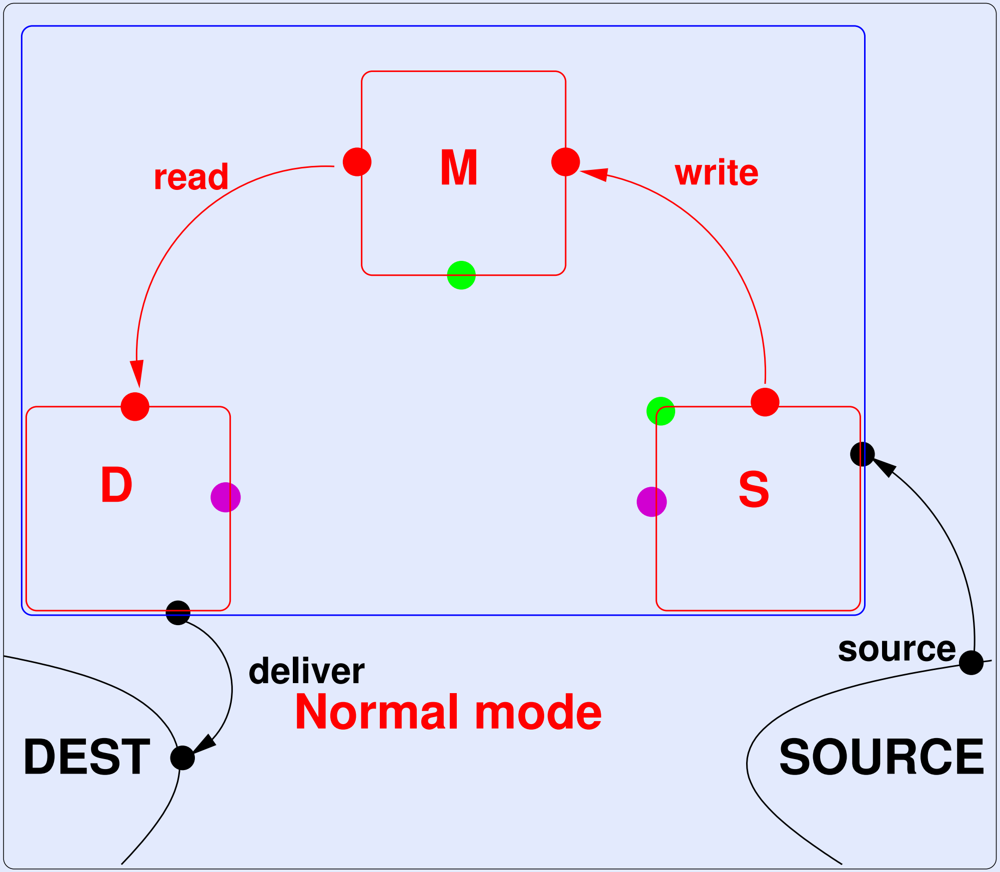


# Example: 3buffer Medium SPEC

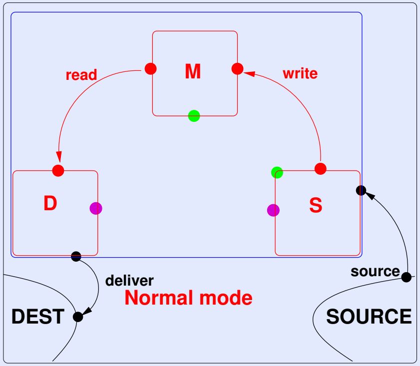



$$
\begin{aligned}
MEDIUM(\varepsilon) &= source(d).MEDIUM(d) \\
MEDIUM(d) &= source(m).MEDIUM(dm) + \\
&\quad \overline{deliver(d)}.MEDIUM(\varepsilon) \\
MEDIUM(dm) &= source(s).MEDIUM(dms) + \\
&\quad \overline{deliver(d)}.MEDIUM(m) \\
MEDIUM(dms) &= \overline{deliver(d)}.MEDIUM(ms)
\end{aligned}
$$

# Example: 3buffer Medium

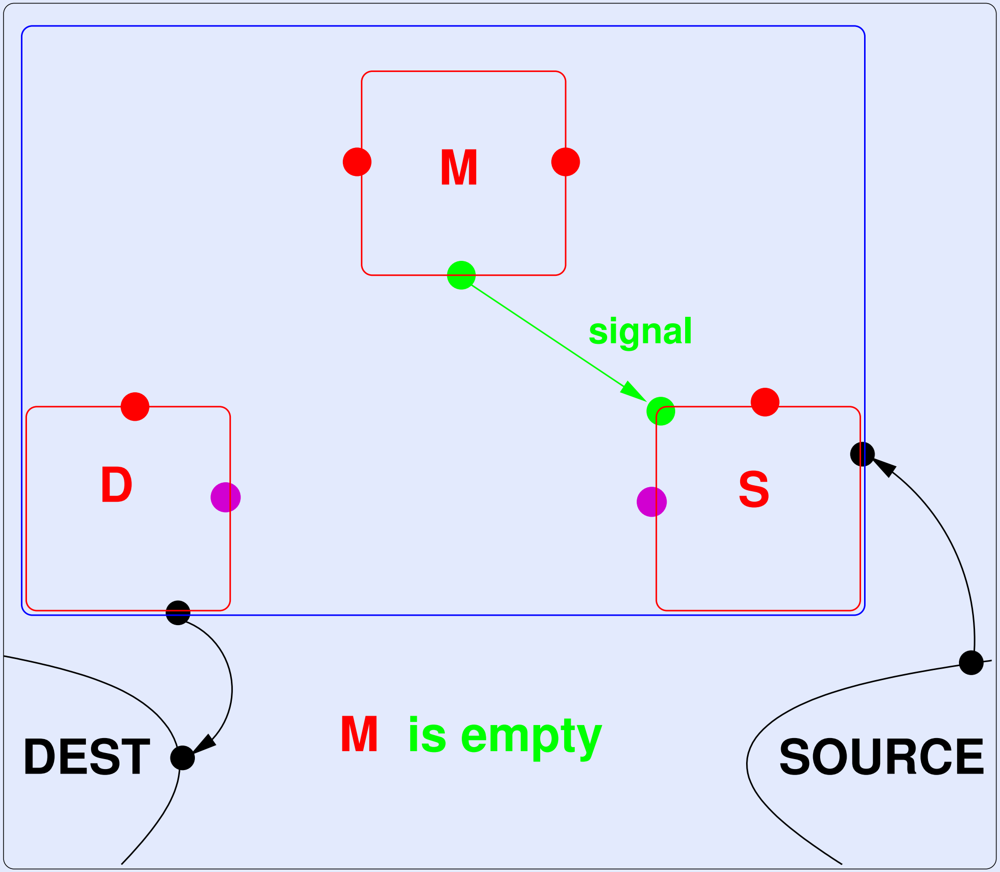


# Example: 3buffer Medium $N$

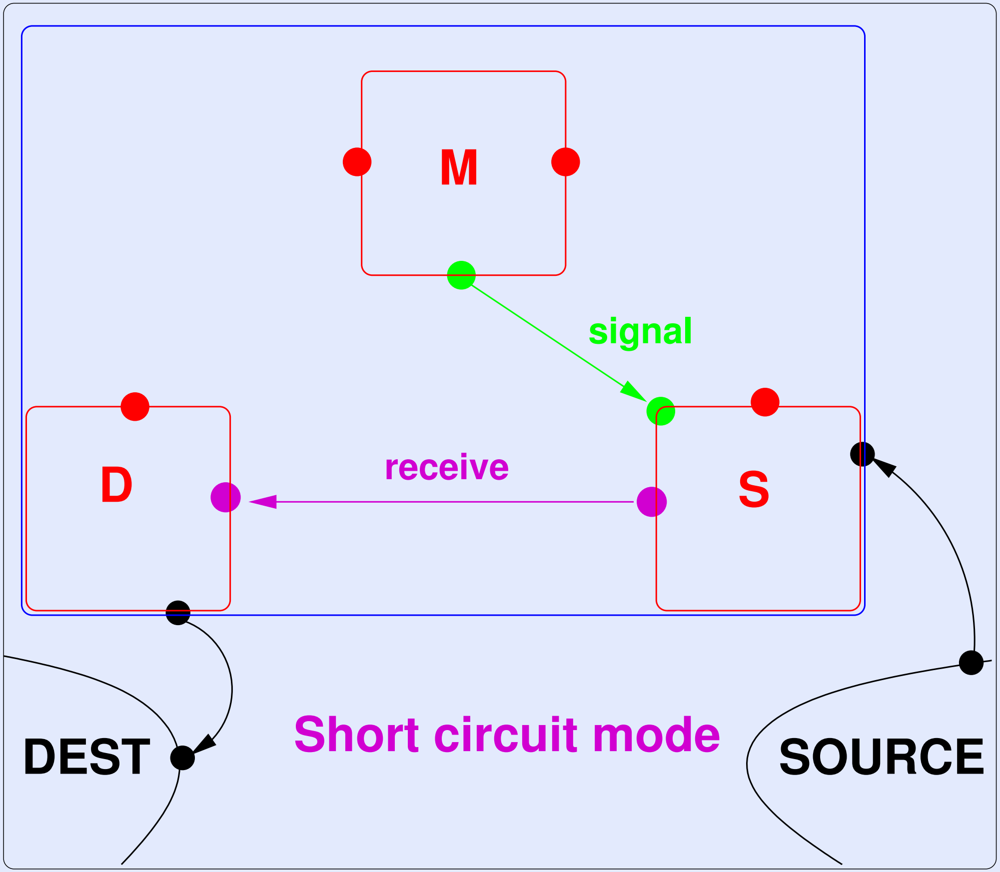



$$
\begin{array}{lcl}
C(\perp) & = & in(u).C(u) \\
C(u) & = & \overline{u}.C(\perp) \\
D_1(u_\perp) & = & C[read/in, deliver/out] \\
M_1(u_\perp) & = & C[read/out, write/in] \\
S_1(u_\perp) & = & C[source/in, write/out] \\
N(d, m, s) & = & (D_1(d) \mid M_1(m) \mid S_1(s)) \backslash \{read, write\}
\end{array}
$$

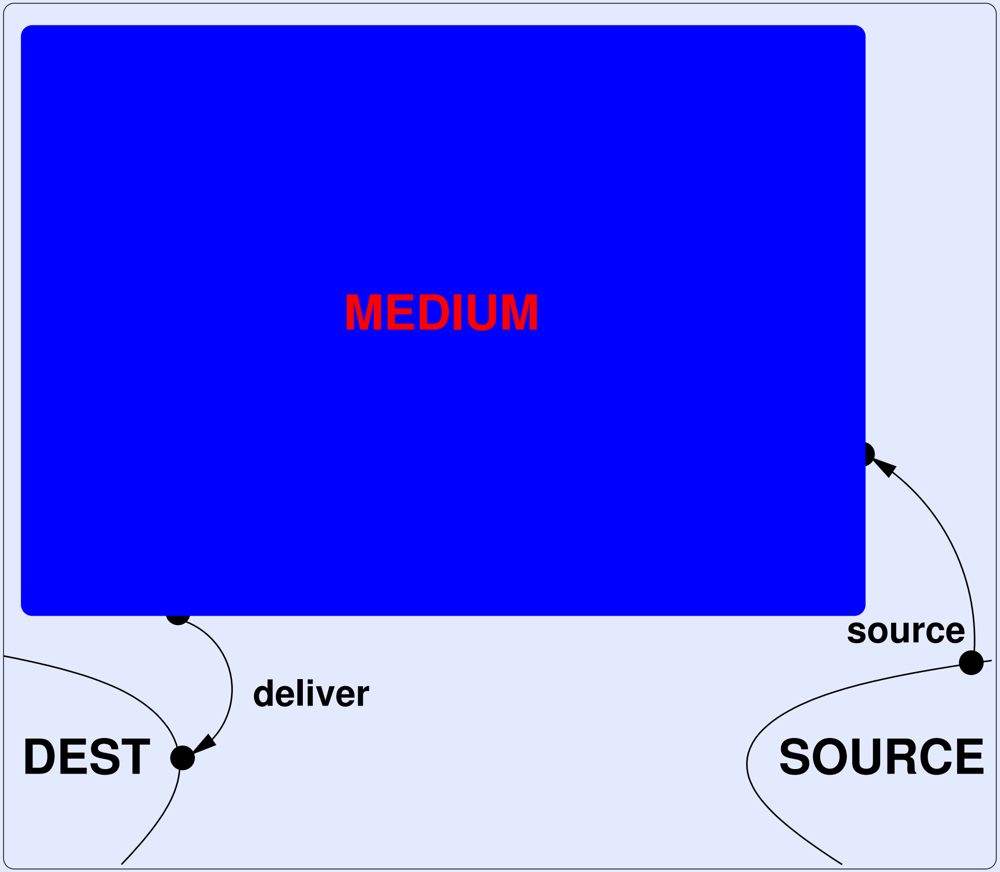
# Example: 3buffer Medium



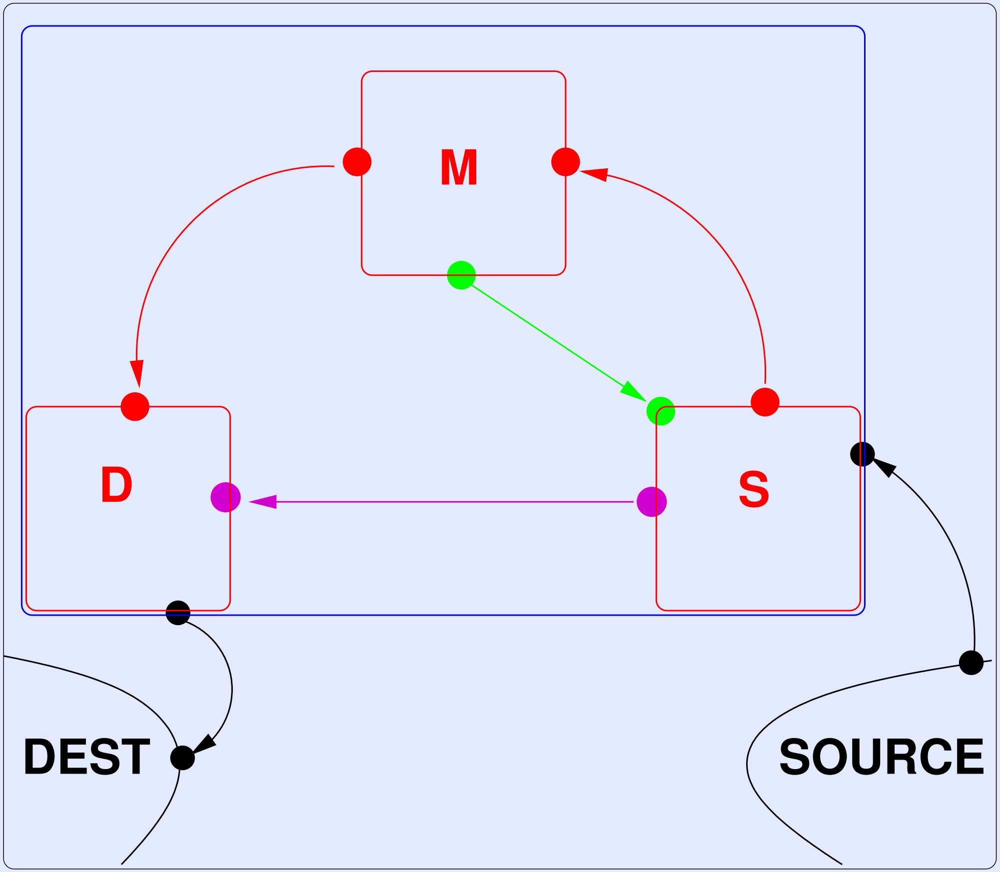
# Example: 3buffer Medium



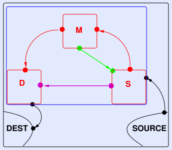
# Example: 3buffer Medium



# Example: 3buffer Medium



# Example: 3buffer Medium $SC$




$$
\begin{array}{lcl}
D_2(\bot) & = & read(u).D_2(u) + receive(v).D(v) \\
D_2(w) & = & \overline{deliver(w)}.D(\bot) \\
M_2(\bot) & = & write(u).M_2(u) \\
M_2(u) & = & \overline{read(u)}.\overline{mempty()}.M(\bot) \\
S_2(\bot, 0) & = & source(u).S_2(u, 0) + mempty().S_2(\bot, 1) \\
S_2(u, 0) & = & mempty().S_2(u, 1) \\
S_2(u, 1) & = & \overline{receive(u)}.S_2(\bot, 1) + \overline{write(u)}.S_2(\bot, 0) \\
S_2(\bot, 1) & = & source(u).S_2(u, 1) \\
Internals & = & \{read, write, receive, mempty\} \\
SC(d, m, s, 1) & = & (D_2(d) \mid M_2(m) \mid S_2(s, 1)) \backslash Internals
\end{array}
$$

# Elaborations

- $R \subseteq \mathbb{P} \times \mathbb{P}$ is a weak bisimulation iff for every $\langle p, q \rangle \in R$ and $a \in Act$,

$$p \xrightarrow{a} p' \Rightarrow \exists q' : q \stackrel{\hat{a}}{\Longrightarrow} q' \wedge p'Rq' \quad and$$
$$q \xrightarrow{a} q' \Rightarrow \exists p' : p \stackrel{\hat{a}}{\Longrightarrow} p' \wedge p'Rq'$$

- $R \subseteq \mathbb{P} \times \mathbb{P}$ is an elaboration[2] if for every $\langle p, q \rangle \in R$ and $a \in Act$,

$$\boxed{p \xrightarrow{a} p' \Rightarrow \exists q' : q \stackrel{\hat{a}}{\Longrightarrow} q' \wedge p'Rq'}$$
$$\boxed{q \xrightarrow{a} q' \Rightarrow \exists p' : p \stackrel{a}{\Longrightarrow} p' \wedge p'Rq'}$$

- $p \lessapprox q$ if $\langle p, q \rangle$ belongs to an elaboration
- $\lessapprox$ is a preorder close to $\approx$ but *finer*.
- $$\begin{array}{lllcccc} \tau.p & \approx & p & \text{and of course,} & p & \approx & \tau.p \\ \tau.p & \lessapprox & p & \text{but} & p & \not\lessapprox & \tau.p \end{array}$$

3buffer

# Efficiency Prebisimulations[1]

- $R \subseteq P \times P$ is an efficiency prebisimulation (EP for short) if for every $\langle p, q \rangle \in R$ the following conditions are satisfied.

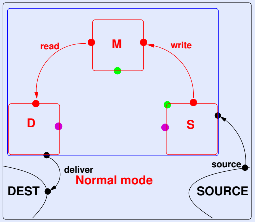
$$p \xrightarrow{\alpha} p' \Rightarrow \exists q' : q \xrightarrow{\alpha} q' \wedge p' R q'$$

$$p \xrightarrow{\tau} p' \Rightarrow p' R q' \vee \exists q' : q \xrightarrow{\tau} q' \wedge p' R q'$$

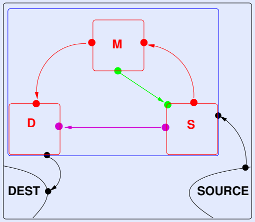
$$q \xrightarrow{a} q' \Rightarrow \exists p' : p \stackrel{a}{\Longrightarrow} p' \wedge p' R q'$$

- $p \lesssim q$ if $\langle p, q \rangle$ belongs to an EP.
- $\lesssim$ is a preorder lying between $\sim$ and $\lessapprox$.

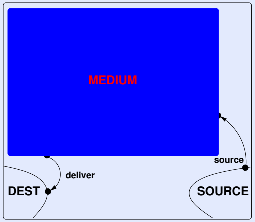
# Example: 3buffer Comparisons




$\lesssim$




$\lesssim$




$$
\begin{array}{lcl}
N(dms) & \approx & SC(dms?) \\
SC(dms?) & \approx & MEDIUM(dms) \\
N(\perp\perp\perp) & \lessapprox & SC(\perp\perp\perp\ 1) \\
SC(\perp\perp\perp\ 1) & \lesssim & MEDIUM(\varepsilon)
\end{array}
$$

$$
N(\perp\perp\perp) \quad \lesssim \quad MEDIUM(\varepsilon)
$$

- All of them are weakly bisimilar to each other,
- But $SC$ can *sometimes* be “quicker” than $N$.
- $MEDIUM$ would be the “most efficient” in *every* run! *Pity it is not available as a basic process!*

Home Page
ICTAC-2004
Page 15 of 37
Go Back
Full Screen
Close
Quit

# LTS

- $V$ : countable set of visible actions, $\alpha \in V$
- $\tau \notin V$: a distinguished invisible action
- $Act = V \cup \{\tau\}$: the set of *actions*, $a \in Act$
- $\langle \mathbb{P}, Act, \longrightarrow \rangle$: A labelled transition system (LTS) where
  - $\mathbb{P}$: a set of process states or processes
  - $\longrightarrow \subseteq \mathbb{P} \times Act \times \mathbb{P}$ is the transition relation.
  - $p \xrightarrow{a} q$: denotes $(p, a, q) \in \longrightarrow$ and
  - $q$ is a strong $a$-derivative of $p$.

# Strong Simulations and $HML^-$

- $R \subseteq \mathbb{P} \times \mathbb{P}$ is a strong simulation (SS) if for every $\langle p, q \rangle \in R, a \in Act$,

$$p \stackrel{a}{\longrightarrow} p' \Rightarrow \exists q' : q \stackrel{a}{\longrightarrow} q' \wedge p'Rq'$$

- $p \leq_{SS} q$ iff $pRq$ for some strong simulation $R$.
- $\leq_{SS}$ is a preorder (reflexive and transitive).

The class $L_{SS}$ of strong simulation formulae over $Act$:

$$\varphi ::= \ <a> \varphi \mid \bigwedge_{i \in I} \varphi_i$$

Note:

- index set $I$ may be infinite.
- $tt$ is the empty conjunction.

# $L_{SS}$: Semantics

The satisfaction relation $\models \ \subseteq \ \mathbb{P} \times L_{SS}$

- $p \models tt$ for all $p \in \mathbb{P}$.
- $p \models <a> \varphi$ for $a \in Act$ if $\exists p' \in \mathbb{P} : p \xrightarrow{a} p'$ and $p' \models \varphi$.
- $p \models \bigwedge_{i \in I} \varphi_i$ if $p \models \varphi_i$ for all $i \in I$.

Alternatively the meaning of a formula is the set of processes that satisfy it:

$$
\begin{aligned}
[\![tt]\!] &= \mathbb{P} \\
[\![<a> \varphi]\!] &= \{p \mid \exists p' : p \xrightarrow{a} p', p' \in [\![\varphi]\!]\} \\
[\![\bigwedge_{i \in I} \varphi_i]\!] &= \bigcap_{i \in I} [\![\varphi_i]\!]
\end{aligned}
$$

# $L_{SS}$: Characterization

**Proposition 1**

$$p \models \varphi \iff p \in [\![\varphi]\!]$$

□

- Consider the set of all formulae a process satisfies,

$$SS(p) = \{\varphi \in L_{SS} \mid p \models \varphi\}$$

- and define an ordering $\sqsubseteq_{SS}$ (induced by $SS(p)$) on processes:

$$p \sqsubseteq_{SS} q \text{ iff } SS(p) \subseteq SS(q)$$

- whose kernel $=_{SS}$, is

$$p =_{SS} q \text{ iff } SS(p) = SS(q)$$

# $L_{SS}$: Characterization

$L_{SS}$ characterizes the behavioural preorder $\leq_{SS}$.

**Theorem 1** *(Hennessy-Milner[3], van Glabeek[4]).*

$$p \leq_{SS} q \iff p \sqsubseteq_{SS} q \iff SS(p) \subseteq SS(q)$$

**Proof.**

($\Rightarrow$). Assume $p \leq_{SS} q$ and $\varphi \in SS(p)$. Proceed by induction on the structure of $\varphi$.

($\Leftarrow$). Show that $\sqsubseteq_{SS}$ is a strong simulation.

$\square$

# Principle of Containment

- Containment. For any behavioural preorder $\leq_B$, a logic $L_B$ characterizes $\leq_B$ if

$$\boxed{p \leq_B q \text{ iff } B(p) \subseteq B(q)}$$

  where $B(p)$ is the set of formulae that $p$ satisfies.

- Consequently, the kernel $=_B$, of the preorder $\leq_B$,

$$=_B \;=\; \leq_B \cap \leq_B{}^{-1}$$

  is an equivalence relation and is characterized by

$$\boxed{p =_B q \text{ iff } B(p) = B(q)}$$

  equality on sets of satisfying formulae.

# Expressiveness

- A logic $L_1$ is as expressive as another $L_2$ if it can express every property that $L_2$ can. It is more expressive, denoted $L_1 \prec L_2$, if there are formulae in $L_1$ that cannot be expressed in $L_2$.
- Given behavioural preorders $\leq_1$, and $\leq_2$ characterized respectively by logics $L_1$ and $L_2$,

$$\leq_1 \ \subset \ \leq_2 \text{ iff } L_1 \prec L_2$$

  since $L_1$ can allow for *finer* distinctions to be made than $L_2$.

# Strong Bisimulations and $HML$

- $R \subseteq \mathbb{P} \times \mathbb{P}$ is a strong bisimulation (SB) if both $R$ and $R^{-1}$ are strong simulations.
- $p \sim q$ iff $pRq$ for some strong bisimulation $R$.
- $\sim$ is an equivalence relation.

The class $L_{SB}$ of strong bisimulation formulae over $Act$:

$$\varphi ::= \; <a> \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \boxed{\neg \varphi}$$

c.f. $L_{SS}$

# $L_{SB}$: Semantics

- As for $L_{SS}$. In addition
- $p \models \neg\varphi$ if $p \not\models \varphi$. Alternatively, $[\![\neg\varphi]\!] = \mathbb{P} - [\![\varphi]\!]$.
- $SB(p)$ is the set of all formulae of HML that $p$ satisfies.
- $p \sqsubseteq_{SB} q$ iff $SB(p) \subseteq SB(q)$ and $p =_{SB} q$ iff $SB(p) = SB(q)$.

**Proposition 2** *(van Glabeek)[4].* $p \sqsubseteq_{SB} q \iff p =_{SB} q$.

**Proof.** If $\varphi \in SB(q) - SB(p)$ then $\neg\varphi \in SB(p) - SB(q)$ which contradicts $p \sqsubseteq_{SB} q$. $\square$

# $L_{SB}$: Characterization

**Theorem 2** *(Hennessy-Milner[3], van Glabeek[4]).*

$$p \sim q \iff p \sqsubseteq_{SB} q \iff SB(p) \subseteq SB(q)$$

**Proof.**

($\Rightarrow$). Assume $p \sim q$ and $\varphi \in SB(p)$. Proceed by induction on the structure of $\varphi$.

($\Leftarrow$). Suffices to show that $=_{SB}$ is a strong bisimulation. By proposition 2 suffices to show that $\sqsubseteq_{SB}$ is a strong simulation since $=_{SB} = \sqsubseteq_{SB} = \sqsubseteq_{SB}{}^{-1}$.

□

# Derived Operators

- Duals:

$$\begin{aligned} [a]\varphi &\equiv \neg < a > \neg\varphi \\ \bigvee_{i\in I}\varphi_i &\equiv \neg \bigwedge_{i\in I} \neg\varphi_i \end{aligned}$$

- Iteration:

$$\begin{aligned} < a >^{0} \varphi &\equiv \varphi \\ < a >^{m+1} \varphi &\equiv < a >< a >^{m} \varphi \quad m \in \mathbb{N} \end{aligned}$$

- Weak Operators:

$$\begin{aligned} \ll\gg\varphi &\equiv \bigvee_{m\geq 0} < \tau >^{m} \varphi \\ \ll a \gg\varphi &\equiv \ll\gg< a >\ll\gg \varphi \end{aligned}$$

- Duals:

$$\begin{aligned} [[\ ]]\varphi &\equiv \neg \ll\gg \neg\varphi \\ [[a]]\varphi &\equiv [[\ ]][a][[\ ]]\varphi \end{aligned}$$

$\ll \hat{a} \gg$ denotes $\ll\gg$ if $a = \tau$ and $\ll a \gg$ otherwise.

Home Page
ICTAC-2004
Page 26 of 37
Go Back
Full Screen
Close
Quit

# Negation-free HML

- Negation may be pushed inward and finally eliminated altogether, provided duals are added to the language.
- The following negation-free logic $L'_{SB}$ is *as expressive* as $L_{SB}$.

$$\varphi ::= \; <a> \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \boxed{[a]\varphi} \;\mid \boxed{\bigvee_{i \in I} \varphi_i}$$

- With *ff* the empty disjunction, $\models$ is extended in the obvious fashion.
  - $p \models ff$ for no process $p$,
  - $p \models [a]\varphi$ for $a \in Act$ if $\forall p' \in \mathbb{P} : p \xrightarrow{a} p' \Rightarrow p' \models \varphi$.
  - $p \models \bigvee_{i \in I} \varphi_i$ if $\exists i \in I : p \models \varphi_i$.

# Weak Bisimulations

- $p \Longrightarrow p$ for all processes $p$
- $p \xrightarrow{\tau} q$ and $q \Longrightarrow r$ implies $p \Longrightarrow r$
- $p \overset{a}{\Longrightarrow} q$ if $p \Longrightarrow \xrightarrow{a} \Longrightarrow q$, and
- $p \overset{\hat{a}}{\Longrightarrow} q$ denotes $p \Longrightarrow q$ if $a = \tau$ and $p \overset{a}{\Longrightarrow} q$ otherwise.
- $R \subseteq \mathbb{P} \times \mathbb{P}$ is a weak simulation if for every $\langle p, q \rangle \in R$ and $a \in Act$,

$$p \xrightarrow{a} p' \Rightarrow \exists q' : q \overset{\hat{a}}{\Longrightarrow} q' \wedge p' R q'$$

  $R$ is a weak bisimulation (WB) if both $R$ and $R^{-1}$ are weak simulations.

- $p \approx q$ if $\langle p, q \rangle$ belongs to a weak bisimulation $R$
- $\approx$ is an equivalence relation.

# Observable HML

- The logic $L_{WB}$ of weak bisimulation formulae over $V$.

$$\varphi \ ::= \boxed{\ll\alpha\gg \varphi} \ \mid \bigwedge_{i\in I} \varphi_i \mid \neg\varphi$$

- $\approx$ is coarser than $\sim$, so it requires a less expressive logic to characterize it.
- The semantics of $\ll\alpha\gg \varphi$ may be derived from $HML$ and $\stackrel{\alpha}{\Longrightarrow}$

$$\boxed{p \models \ll\alpha\gg \varphi \text{ if } \exists p' : p \stackrel{\alpha}{\Longrightarrow} p' \wedge p' \models \varphi}$$

- $WB(p) = \{\varphi \in L_{WB} \mid p \models \varphi\}$.
- $p \sqsubseteq_{WB} q$ iff $WB(p) \subseteq WB(q)$
- $p =_{WB} q$ iff $WB(p) = WB(q)$.

**Theorem 3** *$L_{WB}$ characterizes weak bisimilarity i.e. $p \approx q$ iff $p =_{WB} q$.*

# The Agenda

We require a gradation of logics

- with the following relationships:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| Granularity | $\sim$ | $\subset$ $\lesssim$ | | $\subset$ $\lessapprox$ | | $\subset$ $\approx$ | |
| Expressiveness | $L_{SB}$ | $\prec$ | $L_{EP}$ | $\prec$ | $L_E$ | $\prec$ | $L_{WB}$ |

- and satisfying the principle of containment:

$$
\begin{array}{lllll}
p \sim q & \iff & L_{SB}(p) & = & L_{SB}(q) \\
p \lesssim q & \iff & L_{EP}(p) & \subseteq & L_{EP}(q) \\
p \lessapprox q & \iff & L_{E}(p) & \subseteq & L_{E}(q) \\
p \approx q & \iff & L_{WB}(p) & = & L_{WB}(q)
\end{array}
$$

# Logic for Elaborations

- The class $L_E \supset L_{WB}$ of elaboration formulae over $Act$ is given by the following two-level grammar, where $\alpha \in V$ and $a \in A$.

$$\varphi \quad ::= \quad \ll\alpha\gg \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \neg\varphi$$

$$\pi \quad ::= \quad \varphi \mid \ll\hat{a}\gg \pi \mid \boxed{\epsilon^k \pi} \mid [[a]]\pi \mid \bigwedge_{i \in I} \pi_i \mid \bigvee_{i \in I} \pi_i$$

where $k > 0$.

- Semantics:

$$\boxed{p \models \epsilon^k \pi \text{ if } \forall p' \in \mathbb{P} : p \xrightarrow{\tau^j} p' \wedge p' \models \pi \Rightarrow j < k}$$

More perspicuously,

$$\boxed{p \models \epsilon^k \pi \text{ if } \forall p' \in \mathbb{P} : p \xrightarrow{\tau^j} p' \wedge j \geq k \Rightarrow p' \not\models \pi}$$

# Expressiveness of $L_E$: Examples

- $\epsilon^1 tt \equiv p$ *is stable* i.e. $p$ cannot perform a $\tau$ action.
- $\bigvee_{k>0} \epsilon^k tt \equiv p$ *converges* i.e. $p$ cannot perform an infinite sequence of $\tau$ actions.
- $\epsilon^k ff \equiv p$ *may perform any finite sequence of* $\tau$*'s*. (Caution! Does not necessarily imply that $p$ *diverges*, unless $p$ is also finitely branching).
- $\epsilon^j \pi \Rightarrow \epsilon^k \pi$ for all $0 < j < k$.
- $\bigwedge_{k \geq j} \epsilon^k \pi \iff \epsilon^j \pi$, $j > 0$.
- $p$ *diverges* is not expressible
- Statements specifying lower bounds on number of consecutive $\tau$ actions not expressible.

# Logic of EP

- The class $L_{EP}$ of efficiency prebisimulation formulae over $Act$ is given by the following grammar.

$$\varphi \ ::= \ \ll \alpha \gg \varphi \mid \bigwedge_{i \in I} \varphi_i \mid \neg \varphi$$

$$\pi \ ::= \ \varphi \mid \boxed{< \alpha > \pi} \mid \boxed{(\tau)\pi} \mid \epsilon^k \pi \mid [[a]]\pi$$

$$\mid \bigwedge_{i \in I} \pi_i \mid \bigvee_{i \in I} \pi_i$$

c.f. $L_E$

- Semantics:

$$\boxed{p \models (\tau)\pi \text{ if } p \models \pi \vee (\exists p' \in \mathbb{P} : p \xrightarrow{\tau} p' \wedge p' \models \pi)}$$

# Expressiveness of $L_{EP}$

- The use of $<\alpha>$ instead of $\ll \hat{a} \gg$ obviously adds more expressive power in terms of determining that a visible action is immediately possible.
- The possiblity operator $(\tau)$ does not allow for exact determination of the number of consecutive $\tau$ actions possible.
- Statements such as $p$ *has a* $\tau^2$ *derivative that satisfies* $\pi$ are not expressible.
- HML formulae such as $\neg <\tau><\tau> \varphi$ are not expressible.

# The Characterizations

**Lemma 1** $p \lessapprox q \Rightarrow p =_{E_{WB}} q$ *and hence* $p \lessapprox q \Rightarrow p \sqsubseteq_{E_{WB}} q$.

□

**Theorem 4** $p \lessapprox q$ *iff* $p \sqsubseteq_{E} q$.

□

**Lemma 2** $p \lesssim q \Rightarrow p =_{EP_{WB}} q$ *and hence* $p \lesssim q \Rightarrow p \sqsubseteq_{EP_{WB}} q$.

□

**Theorem 5** $p \lesssim q$ *iff* $p \sqsubseteq_{EP} q$.

□

- Nothing special in the proof techniques used here.
- Proof proceeds in two levels. The proof for the $\pi$ level of the language requires the use of the preceding lemma for the $\varphi$.

# Conclusion & Open Questions

- *Equivalences* have been done before, but *preorders* rarely.
- The principle of containment generalizes from equivalences to preorders very naturally.
- Can something akin to characteristic formulae of Kim Larsen be done in the setting of preorders?
- Solving equations has been done, but how does one solve preordering equations?

# References

[1] S. Arun-Kumar and M.Hennessy. An efficiency preorder for processes. *Acta Informatica, Springer-Verlag*, 1992.

[2] S. Arun-Kumar and V. Natarajan. Conformance: A precongruence close to bisimilarity. In *Proceedings, Structures in Concurrency Theory, Springer Workshops in Computer Science Series*, 1995.

[3] M. Hennessy and R. Milner. Algebraic laws for nondeterminism and concurrency. *Journal of the ACM*, 32:137–161, 1985.

[4] R. J. van Glabeek. The linear time – branching time spectrum I. The semantics of concrete, sequential processes. In *Handbook of Process Algebra*. Elsevier Science B. V., Netherlands., 2001.