Programming Languages

S. Arun-Kumar
Department of Computer Science and Engineering
I. I. T. Delhi, Hauz Khas, New Delhi 110 016.

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1. The Programming Languages Overview
Programs: Source to Runs-2

Source → Compiler/Interpreter → IR

IR → Linked IR → Loader

Linked IR → Linked IR

Linked IR → Loader

Loader → Assembly code → Assembler

Assembler → Machine code

Machine code → Assembler

Assembling Errors

Assembler → Assembling Errors

Assembler → Assembler

Pre-processor

Pre-processing Errors

Pre-processor → Macro-processor

Macro-processor

Translated Source

Translated Source → Macro-translation Errors

Pre-processor

Pre-processing Errors

Compilation Errors

Linking Errors

Loading Errors

Runtime Errors

Results

Runtime System

Runtime System → Runtime Errors

Runtime System

Results
Programs: Source to Runs: \LaTeX

TeX Source

\LaTeX Compiler/Interpreter

IR

TeX

Pre-processing Errors

noweave

Pre-processor

notangle

LaTeX Source

Macros–translation Errors

LaTeX Macro–processor

Target printer code

Postscript Processor

dvi2pdf

dvi2ps

pdf

Display or Printer

Render

Output

Print

Compiler/Interpreter

Linker

Sources to Runs: \LaTeX

Compilation Errors

Linking Errors

Pre-processing Errors

Source Program

LaTeX Source

Macros–translation Errors

LaTeX Macro–processor

Translation TeX Source

\LaTeX dvi

dvi2pdf

dvi2ps

ps

pdf

or

Display

Printer Output

Postscript

Processor

Target

printer
code

Print

Render
Programs: Source to Runs: \LaTeX

Source Program

LaTeX Source

Pre-processing Errors

noweave
Pre-processor
notangle

Macro-translation Errors

Now source
latex source

LaTeX
Macro-processor

Compilation Errors

IR

Tex

Compiler/Interpreter

Linker

Linked IR

Linking Errors

dvi

Postscript Processor

dvi2pdf
dvi2ps

ps

Printer

Target printer code

Printer

Output

Display

Render

Postscript Processor

Tex dvi
dvi2ps
dvi2pdf

pdf

or

Display

Printer

Output

Printer

Display

Render
Programs: Source to Runs: $\LaTeX$

Errors
The Landscape of General PLs
The Usage of General PLs
The Major Features of General PLs
FORTRAN

• The very first high-level programming language
• Still used in scientific computation
• Static memory allocation
• Very highly compute oriented
• Runs very fast because of static memory allocation
• Parameter passing by reference
COBOL

• A business oriented language
• Extremely verbose
• Very highly input-oriented
• Meant to manage large amounts of data on disks and tapes and generate reports
• Not computationally friendly
LisP

• First functional programming language
• Introduced lists and list-operations as the only data-structure
• Introduced symbolic computation
• Much favoured for AI and NLP programming for more than 40 years
• The first programming language whose interpreter could be written in itself.
ALGOL-60

• Introduced the Backus-Naur Form (BNF) for specifying syntax of a programming language
• Formal syntax defined by BNF (an extension of context-free grammars)
• First language to implement recursion
• Introduction of block-structure and nested scoping
• Dynamic memory allocation
• Introduced the call-by-name parameter mechanism
Pascal

• ALGOL-like language meant for teaching structured programming
• Introduction of new data structures – records, enumerated types, subrange types, recursive data-types
• Its simplicity led to its “dialects” being adopted for expressing algorithms in pseudo-code
• First language to be ported across a variety of hardware and OS platforms – introduced the concepts of virtual machine and intermediate code (bytecode)
ML

- First strongly and statically typed functional programming language
- Created the notion of an inductively defined type to construct complex types
- Powerful pattern matching facilities on complex data-types.
- Introduced type-inference, thus making declarations unnecessary except in special cases
- Its module facility is inspired by the algebraic theory of abstract data types
- The first language to introduce functorial programming between algebraic structures and modules
Prolog

• First Declarative programming language
• Uses the Horn clause subset of first-order logic
• Goal-oriented programming implementing a top-down methodology
• Implements backtracking as a language feature
• Powerful pattern-matching facilities like in functional programming
• Various dialects implement various other features such as constraint programming, higher-order functions etc.
2. Introduction to Compiling
Introduction to Compiling

• Translation of programming languages into executable code
• But more generally any large piece of software requires the use of compiling techniques.
• The processes and techniques of designing compilers is useful in designing most large pieces of software.
• Compiler design uses techniques from theory, data structures and algorithms.
Software Examples

Some examples of other software that use compiling techniques

• Almost all user-interfaces require scanners and parsers to be used.
• All XML-based software require interpretation that uses these techniques.
• All mathematical text formatting requires the use of scanning, parsing and code-generation techniques (e.g. \LaTeX).
• Model-checking and verification software are based on compiling techniques
• Synthesis of hardware circuits requires a description language and the final code that is generated is an implementation either at the register-transfer level or gate-level design.
Books and References


Source and Target

In general a compiler for a source language $S$ written in some language $C$ translates code to a target language $T$.

Source $S$ could be

- a programming language, or
- a description language (e.g. Verilog, VHDL), or
- a markup language (e.g. XML, HTML, SGML, LaTeX)

Target $T$ could be

- another programming language, assembly language or machine language, or
- a language for describing various objects (circuits etc.), or
- a low level language for execution, display, rendering etc.

We will be primarily concerned with compiling from a high-level programming language (source) to low-level code.
The Compiling Process

In general the process of compiling involves at least three languages:

1. The language $S$ of source programs in which the users of the compiler write code.

2. The language $C$ in which the compiler itself is written. The assumption is that unless the compiler itself is written in machine language there is already a compiler or an interpreter for $C$.

3. The language $T$ into which the compiler translates the user programs.

Besides these three languages there could be several other intermediate languages $I_1, I_2, \ldots$ (also called intermediate representations) into which the source program could be translated in the process of compiling or interpreting the source programs written in $S$. In modern compilers, for portability, modularity and reasons of code improvement, there is usually at least one intermediate representation.
Compiling as Translation

Except in the case of a source to source translation (for example, a Pascal to C translator which translates Pascal programs into C programs), we may think of the process of compiling high-level languages as one of transforming programs written in $S$ into programs of lower-level languages such as the intermediate representation or the target language. By a low-level language we mean that the language is in many ways closer to the architecture of the target language.
Phases of a Compiler

A compiler or translator is a fairly complex piece of software that needs to be developed in terms of various independent modules. In the case of most programming languages, compilers are designed in phases. The various phases may be different from the various passes in compilation.
The Big Picture: 1

stream of characters

SCANNER

stream of tokens
The Big Picture: 2

The diagram shows a flow of data processing:

1. **Stream of Characters**
2. **Scanner**
3. **Stream of Tokens**
4. **Parser**
5. **Parse Tree**
The Big Picture: 3

SCANNER
stream of characters
stream of tokens
PARSER
parse tree
SEMANTIC ANALYZER
abstract syntax tree
The Big Picture: 4

SCANNER

stream of characters

PARSER

stream of tokens

parse tree

SEMANTIC ANALYZER

abstract syntax tree

I.R. CODE GENERATOR

intermediate representation
The Big Picture: 5

stream of characters

SCANNER

stream of tokens

PARSER

parse tree

SEMANTIC ANALYZER

abstract syntax tree

I.R. CODE GENERATOR

intermediate representation

OPTIMIZER

optimized intermediate representation
The Big Picture: 6

- SCANNER
  - stream of characters
  - stream of tokens
- PARSER
  - parse tree
- SEMANTIC ANALYZER
  - abstract syntax tree
- I.R. CODE GENERATOR
  - intermediate representation
- OPTIMIZER
  - optimized intermediate representation
- CODE GENERATOR
  - target code
The Big Picture: 7

SCANNER

PARSER

SEMANTIC ANALYZER

I.R. CODE GENERATOR

OPTIMIZER

CODE GENERATOR

ERROR-HANDLER

SYMBOL TABLE MANAGER

- stream of characters
- stream of tokens
- parse tree
- abstract syntax tree
- intermediate representation
- optimized intermediate representation
- target code

FLOW CHART
The Big Picture: 8

Scanner | Parser | Semantic Analysis | Symbol Table
IR | Run-time structure |
3. Static Scope Rules
Disjoint Scopes

```plaintext
let
  val x = 10;
  fun fun1 y = let
    ... in ...
    end
in
  fun fun2 z = let
    ... in ...
    end
fun1 (fun2 x)
end
```
Nested Scopes

```plaintext
let
val x = 10;
fun fun1 y =
let
val x = 15
in
x + y
end
end

fun1 x
```
Overlapping Scopes

```plaintext
let
  val x = 10;
  fun fun1 y =
    ...
    ...
    ...
    ...
  fun1 (fun2 x)
end
```
Spannning

let
  val x = 10;
  fun fun1 y = [...]
  fun fun2 z = [...]
end
fun1 (fun2 x)
Scope & Names

• A name may occur either as being defined or as a use of a previously defined name.

• The same name may be used to refer to different objects.

• The use of a name refers to the textually most recent definition in the innermost enclosing scope.
let
  val x = 10; val z = 5;
  fun fun1 y =
    let
      val x = 15
    in
      x + y
    end
end

fun1 x
val x = 10;
fun fun1  y =
let
  val x = 15
in
  x + y * z
end

Back to Scope & Names
let
   val x = 10; val z = 5;
   fun fun1 y =
      let
      val x = 15
      in
      x + y
      end
   end

fun1 x
   val x = 10;
   fun fun1 y =
      let
      val x = 15
      in
      x + y * z
      end
end

Back to Scope & Names
Names & References: 2

```latex
let
val \texttt{x} = 10; val \texttt{z} = 5;
fun \texttt{fun1} \texttt{y} =
    let
        val \texttt{x} = 15
    in
        \texttt{x} + \texttt{y}
    end

val \texttt{z} = 5;
\texttt{z}
end
```

Back to Scope & Names
Names & References: 3

let
val \textcolor{red}{x} = 10; val \textcolor{red}{z} = 5;
fun \textcolor{red}{fun1}
y =
let
val \textcolor{red}{x} = 15
in
\textcolor{red}{x} + y
end
val \textcolor{red}{z} = 5;

Back to Scope & Names
fun1 x
val x = 10;
fun fun1 y =
let
val x = 15
in
x + y
end
val z = 5;
* z
Names & References: 5

```
let
  val x = 10; val z = 5;
  fun fun1
    y = 
    let
      val x = 15
      in
        x + y
      end
    end
  val z = 5;
  * z
end
```

Back to Scope & Names
Names & References: 6

Back to Scope & Names
let
val x = 10; val x = x - 5;
fun fun1 y = let
... in
... end
fun fun2 z = let
... in
... end
fun1 (fun2 x)
in
end
Back to Scope & Names
let
  val x = 10; val x = x - 5;
fun fun1
  y =
  let
    ... 
  in
    ...
  end
fun fun2
  z =
  let
    ... 
  in
    ...
  end
fun1 (fun2 x)
in
end
Names & References: 9

let
val x = 10; val x = x - 5;
fun fun1
  y =
    let
    ...
    in
    ...
    end
  in
func (fun2 x)
in
end

fun fun2
  z =
    let
    ...
    in
    ...
    end
in
fun1 (fun2 x)
in
end

Back to Scope & Names
Definition of Names

Definitions are of the form

\textit{qualifier name} \ldots = \textit{body}

\begin{itemize}
  \item \texttt{val name} =
  \item \texttt{fun name ( argnames )} =
  \item local definitions in definition end
\end{itemize}
Use of Names

Names are used in expressions. Expressions may occur

• by themselves – to be evaluated
• as the \textit{body} of a definition
• as the \textit{body} of a let-expression

\begin{verbatim}
let definitions
in expression
end
\end{verbatim}

use of local
Scope & local

local

fun fun1
\[ y = \ldots \]

fun fun2
\[ z = \ldots \]
  \[ \text{fun1} \]

in

fun fun3
\[ x = \ldots \]
  \[ \text{fun2} \ldots \]
  \[ \text{fun1} \ldots \]

end
4. Runtime Structure
Run-time Structure
Run-time Environment

Memory for running a program is divided up as follows

**Code Segment.** This is where the *object* code of the program resides

**Run-time Stack.** Required in a *dynamic* memory management technique. Especially required in languages which support *recursion*. All data whose sizes can be determined *statically* before loading is stored in an appropriate *stack-frame* (activation record).

**Heap.** All data whose sizes are not determined statically and all data that is generated at run-time is stored in the heap.
A Calling Chain

Main program
  Globals
  Procedure P2
    Locals of P2
      Procedure P21
        Locals of P21
        Body of P21
        Call P21
      Body of P2
      Call P21
  Procedure P1
    Locals of P1
    Body of P1
    Call P2
  Main body
  Call P1

Main → P1 → P2 → P21 → P21
Run-time Structure: 1

Main program

Globals

Procedure P2

Locals of P2

Procedure P21

Locals of P21

Body of P21

Body of P2

Procedure P1

Locals of P1

Body of P1

Main body

Globals

Main
Run-time Structure: 2

Main program

Globals

Procedure P2

Locals of P2

Procedure P21

Locals of P21

Body of P21

Body of P2

Procedure P1

Locals of P1

Body of P1

Main body

Return address to Main

Dynamic link to Main

Locals of P1

Static link to Main

Formal par of P1

Globals

Main → P1
Run-time Structure: 3

Main program

Globals

Procedure P2

Locals of P2

Procedure P21

Locals of P21

Body of P21

Body of P2

Procedure P1

Locals of P1

Body of P1

Main body

Return address to last of P1

Dynamic link to last P1

Locals of P2

Static link to last P1

Formal par P2

Return address to Main

Dynamic link to Main

Locals of P1

Static link to Main

Formal par of P1

Globals

Main → P1 → P2
Run-time Structure: 4

Main program
  Globals
  Procedure P2
    Locals of P2
    Procedure P21
      Locals of P21
      Body of P21
      Body of P2
    Body of P2
  Procedure P1
    Locals of P1
    Body of P1
  Main body

Return address to last of P2
  Dynamic link to last P2
  Locals of P21
  Static link last P2
  Formal par P21

Return address to last of P1
  Dynamic link to last P1
  Locals of P2
  Static link to last P1
  Formal par P2

Return address to Main
  Dynamic link to Main
  Locals of P1
  Static link to Main
  Formal par of P1

Globals

Main → P1 → P2 → P21
Run-time Structure: 5

Main program
- Globals
- Procedure P2
  - Locals of P2
  - Procedure P21
    - Locals of P21
    - Body of P21
  - Body of P2
- Procedure P1
  - Locals of P1
  - Body of P1
  - Main body
- Return address to Main
- Dynamic link to Main
- Locals of P1
- Static link to Main
- Formal par P1

Return address to last of P2
- Dynamic link to last P2
- Locals of P2
- Static link last P2
- Formal par P2

Return address to last of P21
- Dynamic link to last P21
- Locals of P21
- Static link last P2	
- Formal par P21

Main → P1 → P2 → P21 → P21

Back to the Big Picture
5. Scanning or Lexical Analysis

Lexical Analysis
Programming Language Elements

• Every language is built from a finite **alphabet** of **symbols**. The alphabet of a programming language consists of the symbols of the ASCII set.

• Each language has a **vocabulary** consisting of **words**. Each word is a **string of symbols** drawn from the alphabet.

• Each language has a finite set of **punctuation symbols**, which separate phrases, clauses and sentences.

• A programming language also has a finite set of **operators**.

• The phrases, clauses and sentences of a programming language are expressions, commands, functions, procedures and programs.
Lexical Analysis

**lex-i-cal**: relating to words of a language

• A *source program* (usually a file) consists of a stream of characters.

• Given a stream of characters that make up a *source program* the compiler must first break up this stream into groups of meaningful words, and other symbols.

• Each such group of characters is then classified as belonging to a certain *token type*.

• Certain sequences of characters are not tokens and are completely ignored (or skipped) by the compiler.
Tokens and Non-tokens

Tokens Typical tokens are

- **Constants**: Integer, Boolean, Real, Character and String constants.
- **Identifiers**: Names of variables, constants, procedures, functions etc.
- **Keywords/Reserved words**: void, public, main
- **Operators**: +, *, /
- **Punctuation**: , , : ,
- **Brackets**: (, ), [, ], begin, end, case, esac

Non-tokens Typical non-tokens

- **whitespace**: sequences of tabs, spaces, new-line characters,
- **comments**: compiler ignores comments
- **preprocessor directives**: #include ..., #define ...
- **macros** in the beginning of C programs
Scanning: 1

During the scanning phase the compiler/interpreter

• takes a stream of characters and identifies tokens from the lexemes.

• Eliminates comments and redundant whitespace.

• Keeps track of line numbers and column numbers and passes them as parameters to the other phases to enable error-reporting and handling to the user.
Scanning: 2

Definition 5.1 A lexeme is a basic lexical unit of a language consisting of one word or several words, the elements of which do not separately convey the meaning of the whole.

• Whitespace: A sequence of space, tab, newline, carriage-return, form-feed characters etc.
• Lexeme: A sequence of non-whitespace characters delimited by whitespace or special characters (e.g. operators or punctuation symbols)
• Examples of lexemes.
  – reserved words, keywords, identifiers etc.
  – Each comment is usually a single lexeme
  – preprocessor directives
Scanning: 3

Definition 5.2 A token consists of an abstract name and the attributes of a lexeme.

- Token: A sequence of characters to be treated as a single unit.
- Examples of tokens.
  - Reserved words (e.g. `begin`, `end`, `struct`, `if` etc.)
  - Keywords (`integer`, `true` etc.)
  - Operators (`+`, `&&`, `++` etc)
  - Identifiers (variable names, procedure names, parameter names)
  - Literal constants (numeric, string, character constants etc.)
  - Punctuation marks (``, ``, etc.)
Scanning: 4

• Identification of tokens is usually done by a Deterministic Finite-state automaton (DFA).

• The set of tokens of a language is represented by a large regular expression.

• This regular expression is fed to a lexical-analyser generator such as Lex, Flex or JLex.

• A giant DFA is created by the Lexical analyser generator.
Lexical Rules

- Every programming language has **lexical rules** that define how a token is to be defined.

**Example.** In most programming languages identifiers satisfy the following rules.

1. An identifier consists of a sequence of letters (A ... Z, a ... z), digits (0 ... 9) and the underscore (_) character.
2. The first character of an identifier must be a letter.

- Any two tokens are separated by some **delimiters** (usually whitespace) or **non-tokens** in the source program.
5.1. Regular Expressions

Regular Expressions
Specifying Lexical Rules

We require compact and simple ways of specifying the lexical rules of the tokens of a language. In particular,

• there are an *infinite* number of legally correct identifiers (names) in any programming language.

• we require *finite descriptions/specifications* of the lexical rules so that they can cover the infinite number of legal tokens.

One way of specifying the lexical rules of a programming language is to use regular expressions.
Regular Expressions Language

• Each regular expression is a finite sequence of symbols.

• A regular expression may be used to describe an infinite collection of strings.

The regular expression used to define the set of possible identifiers as defined by the rules is

\[ [A−Za−z][A−Za−z0−9_]* \]
Concatenations

Consider a (finite) alphabet (of symbols) \( \mathcal{A} \).

- Any set of strings built up from the symbols of \( \mathcal{A} \) is called a **language**.

- Given any two strings \( x \) and \( y \) in a language, \( x.y \) or simply \( xy \) is the **concatenation of the two strings**.

**Example** Given the strings \( x = \) Mengesha and \( y = \) Mamo, \( x.y = \) MengeshaMamo and \( y.x = \) MamoMengesha.

- Given two languages \( X \) and \( Y \), then \( X.Y \) or simply \( XY \) is the **concatenation of the languages**.

**Example** Let \( X = \{ \text{Mengesha, Gemechis} \} \) and \( Y = \{ \text{Mamo, Bekele, Selassie} \} \)
\( XY = \{ \text{MengeshaMamo, MengeshaBekele, MengeshaSelassie, GemechisMamo, GemechisBekele, GemechisSelassie} \} \)
Note on the Concept of “language”.

Unfortunately we have too many related but slightly different concepts, each of which is simply called a “language”. Here is a clarification of the various concepts that we use.

- Every language has a non-empty finite set of symbols called **letters**. This non-empty finite set is called the **alphabet**.
- Each **word** is a finite sequence of symbols called **letters**.
- The words of a language usually constitute its **vocabulary**. Certain sequences of symbols may not form a word in the vocabulary. A vocabulary for a natural language is defined by a **dictionary**, whereas for a programming language it is usually defined by **formation rules**.

- A **phrase**, **clause** or **sentence** is a finite sequence of words drawn from the vocabulary.
- Every natural language or programming language is a finite or infinite set of **sentences**.
- In the case of formal languages, the formal language is the set of words that can be formed using the formation rules. The language is also said to be **generated** by the formation rules.

There are a variety of languages that we need to get familiar with.

**Natural languages.** These are the usual languages such as **English, Hindi, French, Tamil** which we employ for daily communication and in teaching, reading and writing.

**Programming languages.** These are the languages such as **C, Java, SML, Perl, Python** etc. that are used to write computer programs in.

**Formal languages.** These are languages which are **generated** by certain formation rules.
Meta-languages. These are usually natural languages used to explain concepts related to programming languages or formal languages. We are using English as the meta-language to describe and explain concepts in programming languages and formal languages.

In addition, we do have the concept of a dialect of a natural language or a programming language. For example the natural languages like Hindi, English and French do have several dialects. A dialect (in the case of natural languages) is a particular form of a language which is peculiar to a specific region or social group. Creole (spoken in Mauritius) is a dialect of French, Similarly Brij, Awadhi are dialects of Hindi. A dialect (in the case of programming languages) is a version of the programming language. There are many dialects of C and C++. Similarly SML-NJ and poly-ML are dialects of Standard ML. The notion of a dialect does not really exist for formal languages.

Closer home to what we are discussing, the language of regular expressions is a formal language which describes the rules for forming the words of a programming language. Each regular expression represents a finite or infinite set of words in the vocabulary of a programming language. We may think of the language of regular expressions also as a functional programming language for describing the vocabulary of a programming language. It allows us to generate words belonging to the vocabulary of a programming language

Any formally defined language also defines an algebraic system of operators applied on a carrier set. Every operator in any algebraic system has a pre-defined arity which refers to the number of operands it requires. In the case of regular expressions, the operators are concatenation and alternation are 2-ary operators (binary operators), whereas the Kleene closure and plus closure are 1-ary operators (unary). In addition the letters of the alphabet, which are constants may be considered to be operators of arity 0.
Simple Language of Regular Expressions

We consider a simple language of regular expressions. Assume a (finite) alphabet \( A \) of symbols. Each regular expression \( r \) denotes a set of strings \( \mathcal{L}(r) \). \( \mathcal{L}(r) \) is also called the language specified by the regular expression \( r \).

Symbol For each symbol \( a \) in \( A \), the regular expression \( a \) denotes the set \( \{a\} \).

(Con)catenation For any two regular expressions \( r \) and \( s \), \( rs \) or simply \( rs \) denotes the concatenation of the languages specified by \( r \) and \( s \). That is,

\[
\mathcal{L}(rs) = \mathcal{L}(r)\mathcal{L}(s)
\]
Epsilon and Alternation

Epsilon $\epsilon$ denotes the language with a single element the **empty** string ("") i.e.

$$L(\epsilon) = \{ "\" \}$$

Alternation Given any two regular expressions $r$ and $s$, $r | s$ is the set union of the languages specified by the individual expressions $r$ and $s$ respectively.

$$L(r | s) = L(r) \cup L(s)$$

Example $L(\text{Menelik}|\text{Selassie}|\epsilon) = \{\text{Menelik}, \text{Selassie}, ""\}$. 
String Repetitions

For any string $x$, we may use concatenation to create a string $y$ with as many repetitions of $x$ as we want, by defining repetitions by induction.

$$
\begin{align*}
  x^0 &= \text{""} \\
  x^1 &= x \\
  x^2 &= xx \\
  &\vdots \\
  x^{n+1} &= xx^n = x^n.x \\
  &\vdots 
\end{align*}
$$

Then

$$
x^* = \{ x^n \mid n \geq 0 \}
$$
String Repetitions Example

Example. Let $x = \text{Selassie}$. Then

$$x^0 = \text{""}$$
$$x^1 = \text{Selassie}$$
$$x^2 = \text{SelassieSelassie}$$
$$\vdots$$
$$x^5 = \text{SelassieSelassieSelassieSelassieSelassieSelassie}$$
$$\vdots$$

Then $x^*$ is the language consisting of all strings that are finite repetitions of the string $\text{Selassie}$.
Language Iteration

The * operator can be extended to languages in the same way. For any language $X$, we may use concatenation to create a another language $Y$ with as many repetitions of the strings in $X$ as we want, by defining repetitions by induction.

$$
X^0 = \
X^1 = X
X^2 = X.X
\vdots
X^{n+1} = X.X^n = X^n.X
\vdots
$$

Then

$$
X^* = \bigcup_{n \geq 0} X^n
$$
Language Iteration Example

**Example** Let $X = \{\text{Mengesha}, \text{Gemechis}\}$. Then

\[
X^0 = \{\text{"\text{"}"\text{"}}\}
\]

\[
X^1 = \{\text{Mengesha, Gemechis}\}
\]

\[
X^2 = \{\text{MengeshaMengesha, GemechisMengesha, GemechisGemechis, GemechisGemechisGemechis}\}
\]

\[
X^3 = \{\text{MengeshaMengeshaMengesha, GemechisMengeshaMengesha, GemechisGemechisMengesha, GemechisGemechisGemechisMengesha, GemechisGemechisMengeshaGemechis, GemechisGemechisMengeshaGemechis, GemechisGemechisGemechisGemechis, GemechisGemechisGemechisGemechisGemechis}\}
\]

\[
\vdots
\]

\[
X^{n+1} = X.X^n
\]

\[
\vdots
\]
Kleene Closure

Given a regular expression $r$, $r^n$ specifies the $n$-fold iteration of the language specified by $r$.

Given any regular expression $r$, the Kleene closure of $r$, denoted $r^*$ specifies the language $(\mathcal{L}(r))^*$.

In general

$$r^* = r^0 \mid r^1 \mid \ldots \mid r^{n+1} \mid \ldots$$

denotes an infinite union of languages.

Further it is easy to show the following identities.

$$r^* = \epsilon \mid r.r^* \quad (1)$$

$$r^* = (r^*)_* \quad (2)$$
Plus Closure

The Kleene closure allows for zero or more iterations of a language. The +-closure of a language $X$ denoted by $X^+$ and defined as

$$X^+ = \bigcup_{n>0} X^n$$

denotes one or more iterations of the language $X$.

Analogously we have that $r^+$ specifies the language $(\mathcal{L}(r))^+$. Notice that for any language $X$, $X^+ = X.X^*$ and hence for any regular expression $r$ we have

$$r^+ = r.r^*$$

We also have the identity (1)

$$r^* = \epsilon \mid r^+$$
Range Specifications

We may specify ranges of various kinds as follows.

• \([a-c] = a \mid b \mid c\). Hence the expression of Question 3 may be specified as \([a-c]^*\).

• Multiple ranges: \([a-c0-3] = [a-c] \mid [0-3]\)

Question 6. Try to understand what the regular expression for identifiers really specifies.

Question 7. Modify the regular expression so that all identifiers start only with upper-case letters.

Question 8. Give regular expressions to specify

• real numbers in fixed decimal point notation
• real numbers in floating point notation
• real numbers in both fixed decimal point notation as well as floating point notation.
Equivalence of Regular Expressions

Definition 5.3 Let \( \text{REGEXP}_A \) denote the set of regular expressions over a finite non-empty set of symbols \( A \) and let \( r, s \in \text{REGEXP}_A \). Then

- \( r \leq_A r \) if and only if \( L(r) \subseteq L(s) \) and
- they are equivalent (denoted \( r =_A s \)) if they specify the same language, i.e.

\[
    r =_A s \text{ if and only if } L(r) = L(s)
\]

We have already considered various identities (e.g. (1)) giving the equivalence between different regular expressions.
Notes on bracketing and precedence of operators

In general regular expressions could be ambiguous (in the sense that the same expression may be interpreted to refer to different languages. This is especially so in the presence of

• multiple binary operators
• some unary operators used in prefix form while some others are used in post-fix form. The Kleene-closure and plus closure are operators in postfix form. We have not introduced any prefix unary operator in the language of regular expressions.

All expressions may be made unambiguous by specifying them in a fully parenthesised fashion. However, that leads to too many parentheses and is often hard to read. Usually rules for precedence of operators is defined and we may use the parentheses “(“ and “)” to group expressions over-riding the precedence conventions of the language.

For the operators of regular expressions we will use the precedence convention that \(|\) has a lower precedence than \(\). and that all unary operators have the highest precedence.

Example 5.4 The language of arithmetic expressions over numbers uses the “BDMAS” convention that brackets have the highest precedence, followed by division and multiplication and the operations of addition and subtraction have the lowest precedence.

Example 5.5 The regular expression \(r.s|t.u\) is ambiguous because we do not know beforehand whether it represents \((r.s)|(t.u)\) or \(r.(s|t).u\) or even various other possibilities. By specifying that the operator \(|\) has lower precedence than \(\), we are disambiguating the expression to mean \((r.s)|(t.u)\).

Example 5.6 The language of arithmetic expressions can also be extended to include the unary postfix operation
in which case an expression such as \(-a!\) becomes ambiguous. It could be interpreted to mean either \(-(a!))\) or \(-a!\). In the absence of a well-known convention it is best adopt parenthesisation to disambiguate the expression.

Besides the ambiguity created by multiple binary operators, there are also ambiguities created by the same operator and in deciding in what order two or more occurrences of the same operator need to be evaluated. A classic example is the case of subtraction in arithmetic expressions.

**Example 5.7** The arithmetic expression \(a - b - c\), in the absence of any well-defined convention could be interpreted to mean either \((a - b) - c\) or \(a - (b - c)\) and the two interpretations would yield different values in general. The problem does not exist for operators such addition and multiplication on numbers, because these operators are associative. Hence even though \(a + b + c\) may be interpreted in two different ways, both interpretations yield identical values.

**Example 5.8** Another non-associative operator in arithmetic which often leaves students confused is the exponentiation operator. Consider the arithmetic expression \(a^b^c\). For \(a = 2\), \(b = 3\), \(c = 4\) is this expression to be interpreted as \(a^{(b^c)}\) or as \((a^b)^c\)?

**Exercise 5.1**

1. For what regular expression \(r\) will \(r^*\) specify a finite set?
2. How many strings will be in the language specified by \((a | b | c)^n\)?
3. Give an informal description of the language specified by \((a | b | c)^*\)?
4. Give a regular expression which specifies the language \(\{a^k | k > 100\}\).
5. Simplify the expression \(r^* . r^*\), i.e. give a simpler regular expression which specifies the same language.
6. Simplify the expression \(r^+ . r^+\).
5.2. Nondeterministic Finite Automata (NFA)

Nondeterministic Finite Automata (NFA)
Nondeterministic Finite Automata

A regular expression is useful in defining a finite state automaton. An automaton is a machine (simple program) which can be used to recognize all valid lexical tokens of a language.

A nondeterministic finite automaton (NFA) $N$ over a finite alphabet $A$ consists of

• a finite set $Q$ of states,

• an initial state $q_0 \in Q$,

• a finite subset $F \subseteq Q$ of states called the final states or accepting states, and

• a transition relation $\rightarrow \subseteq Q \times (A \cup \{\varepsilon\}) \times Q$. Equivalently

$\rightarrow: Q \times (A \cup \{\varepsilon\}) \rightarrow 2^Q$

is a function that for each source state $q \in Q$ and symbol $a \in A$ associates a (possibly empty) set of target states.
Nondeterminism and Automata

• In general the automaton *reads* the input string from left to right.
• It reads each input symbol *only once* and executes a transition to new state.
• The $\varepsilon$ transitions represent going to a new target state *without* reading any input symbol.
• The NFA may be nondeterministic because of
  – one or more $\varepsilon$ transitions from the same source state *different* target states,
  – one or more transitions on the *same input* symbol from one source state to two or more different target states,
  – choice between executing a transition on an input symbol and a transition on $\varepsilon$ (and going to different states).
Acceptance of NFA

• For any alphabet $A$, $A^*$ denotes the set of all (finite-length) strings of symbols from $A$.

• Given a string $x = a_1a_2\ldots a_n \in A^*$, an accepting sequence is a sequence of transitions

\[
q_0 \xrightarrow{\varepsilon} \cdots \xrightarrow{a_1} \xrightarrow{\varepsilon} \cdots q_1 \xrightarrow{\varepsilon} \cdots \xrightarrow{a_2} \cdots \xrightarrow{\varepsilon} q_n \xrightarrow{\varepsilon} \cdots
\]

where $q_n \in F$ is an accepting state.

• Since the automaton is nondeterministic, it is also possible that there exists another sequence of transitions

\[
q_0 \xrightarrow{\varepsilon} \cdots \xrightarrow{a_1} \xrightarrow{\varepsilon} \cdots q'_1 \xrightarrow{\varepsilon} \cdots \xrightarrow{a_2} \cdots \xrightarrow{\varepsilon} q'_n \xrightarrow{\varepsilon} \cdots
\]

where $q'_n$ is not a final state.

• The automaton accepts $x$, if there is an accepting sequence for $x$. 
Language of a NFA

• The language accepted or recognized by a NFA is the set of strings that can be accepted by the NFA.

• \( L(N) \) is the language accepted by the NFA \( N \).
Construction of NFAs

• We show how to construct an NFA to accept a certain language of strings from the regular expression specification of the language.

• The method of construction is by *induction on the structure* of the regular expression. That is, for each regular expression operator, we show how to construct the corresponding automaton assuming that the NFAs corresponding to individual components of expression have already been constructed.

• For any regular expression \( r \) the corresponding NFA constructed is denoted \( N_r \). Hence for the regular expression \( r \mid s \), we construct the NFA \( N_{r \mid s} \) using the NFAs \( N_r \) and \( N_s \) as the building blocks.

• Our method requires only one initial state and one final state for each automaton. Hence in the construction of \( N_{r \mid s} \) from \( N_r \) and \( N_s \), the initial states and the final states of \( N_r \) and \( N_s \) are not initial or final unless explicitly used in that fashion.
Constructing NFA

- We show the construction only for the most basic operators on regular expressions.

- For any regular expression \( r \), we construct a NFA \( N_r \) whose initial state is named \( r_0 \) and final state \( r_f \).

- The following symbols show the various components used in the depiction of NFAs.
Regular Expressions to NFAs: 1

We may also express the automaton in tabular form as follows:

<table>
<thead>
<tr>
<th>$N_a$</th>
<th>Input Symbol</th>
</tr>
</thead>
<tbody>
<tr>
<td>State</td>
<td>a</td>
</tr>
<tr>
<td></td>
<td>···</td>
</tr>
<tr>
<td></td>
<td>ε</td>
</tr>
<tr>
<td>$a_0$</td>
<td>{a_f}</td>
</tr>
<tr>
<td></td>
<td>∅ ··· ∅ ∅</td>
</tr>
<tr>
<td>$a_f$</td>
<td>∅</td>
</tr>
<tr>
<td></td>
<td>∅ ··· ∅ ∅</td>
</tr>
</tbody>
</table>

Notice that all the cells except one have empty targets.
### Regular Expressions to NFAs: 2

**Diagram:**

- Initial state: $\varepsilon_0$
- Final state: $\varepsilon_f$
- Transition: $\varepsilon$

<table>
<thead>
<tr>
<th>Transition</th>
<th>Input Symbol</th>
<th>State</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>$a$</td>
<td>$\varepsilon$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_0$</td>
<td>$\emptyset$ $\emptyset$ $\cdots$ $\emptyset$ ${\varepsilon_f}$</td>
</tr>
<tr>
<td></td>
<td>$\varepsilon_f$</td>
<td>$\emptyset$ $\emptyset$ $\cdots$ $\emptyset$ $\emptyset$</td>
</tr>
</tbody>
</table>
Regular Expressions to NFAs: 3

| $N_r|s$ | Input Symbol |
|-------|-------------|
| **State** | a | ⋯ | $\varepsilon$ |
| $r|s_0$ | $\emptyset$ | ⋯ | $\{r_0, s_0\}$ |
| $r_0$ | ⋯ | ⋯ | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $r_f$ | ⋯ | ⋯ | $\{r|s_f\}$ |
| $s_0$ | ⋯ | ⋯ | ⋯ |
| ⋮ | ⋮ | ⋮ | ⋮ |
| $s_f$ | ⋯ | ⋯ | $\{r|s_f\}$ |
| $r|s_f$ | $\emptyset$ | ⋯ | $\emptyset$ |
Regular Expressions to NFAs:4

Notice that the initial state of $N_{r,s}$ is $r_0$ and the final state is $s_f$ in this case.
Regular Expressions to NFAs: 5

\[
\begin{array}{c|c|c|c}
\text{State} & a & \cdots & \varepsilon \\
\hline
r_0^* & \emptyset & \cdots & \{r_0, r_f^*\} \\
\hline
r_0 & \cdots & \cdots & \cdots \\
\hline
\vdots & \vdots & \vdots & \vdots \\
\hline
r_f & \cdots & \cdots & \{r_0, r_f^*\} \\
\hline
r_f^* & \emptyset & \emptyset & \emptyset \\
\end{array}
\]
Regular expressions vs. NFAs

• It is obvious that for each regular expression $r$, the corresponding NFA $N_r$ is correct by construction i.e.

$$\mathcal{L}(N_r) = \mathcal{L}(r)$$

• Each regular expression operator
  – adds at most 2 new states and
  – adds at most 4 new transitions

• Every state of each $N_r$ so constructed has
  – either 1 outgoing transition on a symbol from $A$
  – or at most 2 outgoing transitions on $\varepsilon$

• Hence $N_r$ has at most $2|r|$ states and $4|r|$ transitions.
Example

We construct a NFA for the regular expression \((a|b)^*abb\).

- Assume the alphabet \(A = \{a, b\}\).
- We follow the steps of the construction as given in *Constructing NFA to Regular Expressions to NFAs*:
- For ease of understanding we use the regular expression itself (subscripted by 0 and \(f\) respectively) to name the two new states created by the regular expression operator.
Example:-6

Steps in NFA for \((a|b)^*abb\)
Example:-5

Steps in NFA for $(a|b)^*abb$
Example:-4

Steps in NFA for \((a|b)^*abb\)
Example:-3

Steps in NFA for \((a|b)^*abb\)
Example:-2

Steps in NFA for \((a|b)^*abb\)
Example:- 1

Steps in NFA for \((a \mid b)^* abb\)
Example-final

Steps in NFA for \((a|b)^*abb\)
Extensions

We have provided constructions for only the most basic operators on regular expressions. Here are some extensions you can attempt

1. Show how to construct a NFA for ranges and multiple ranges of symbols

2. Assuming $N_r$ is a NFA for the regular expression $r$, how will you construct the NFA $N_{r^+}$.

3. Certain languages like Perl allow an operator like $r\{k, n\}$, where

$$\mathcal{L}(r\{k, n\}) = \bigcup_{k \leq m \leq n} \mathcal{L}(r^m)$$

Show to construct $N_{r\{k, n\}}$ given $N_r$.

4. Consider a new regular expression operator $^\wedge$ defined by

$$\mathcal{L}(^\wedge r) = A^* - \mathcal{L}(r)$$

What is the automaton $N_{r^\wedge}$ given $N_r$?
Scanning Using NFAs
Scanning and Automata

• **Scanning** is the only phase of the compiler in which every character of the source program is read.

• The scanning phase therefore needs to be defined *accurately* and *efficiently*.

• *Accuracy* is achieved by regular expression specification of the tokens.

• *Efficiency* implies that the input should not be read more than once.
Nondeterminism and Token Recognition

(i) It is difficult to know which $\varepsilon$ transition to pick without reading any further input.

(ii) For two transitions on the same input symbol $a$, it is difficult to know which of them would reach a final state on further input.

(iii) Given an input symbol $a$ and an $\varepsilon$ transition on the current state, it is impossible to decide which one to take without looking at further input.
Nondeterministic Features

- In general it is impossible to recognize tokens in the presence of nondeterminism without *backtracking*.
- Hence NFAs are not directly useful for scanning because of the presence of nondeterminism.
- The nondeterministic feature of the construction of $N_r$ for any regular expression $r$ is in the $\varepsilon$ transitions.
- The $\varepsilon$ transitions in any automaton refer to the fact that no input character is consumed in the transition.
- *Backtracking* usually means algorithms involving them are very complex and hence inefficient.
- To avoid backtracking, the automaton should be made *deterministic*
From NFA to DFA

• Since the only source of nondeterminism in our construction are the $\varepsilon$, we need to eliminate them without changing the language recognized by the automaton.

• Two consecutive $\varepsilon$ transitions are the same as one. In fact any number of $\varepsilon$ transitions are the same as one. So as a first step we compute all finite sequences of $\varepsilon$ transitions and collapse them into a single $\varepsilon$ transition.

• Two states $q, q'$ are equivalent if there are only $\varepsilon$ transitions between them. This is called the $\varepsilon$-closure of states.
Given a set $T$ of states, then $T_\varepsilon = \varepsilon$-closure$(T)$ is the set of states which either belong to $T$ or can be reached from states belonging to $T$ only through a sequence of $\varepsilon$ transitions.

**Algorithm 1 $\varepsilon$-Closure**

**Require:** $T$ a set of states of the NFA  
**Ensure:** $T_\varepsilon = \varepsilon$-CLOSURE$(T)$.  
1. $U := T$  
2. repeat  
3. $U_{old} := U$  
4. $U := U_{old} \cup \{q' \mid q' \notin U, \exists q \in U_{old} : q \xrightarrow{\varepsilon} q'\}$  
5. until $U = U_{old}$  
6. $T_\varepsilon = U$  
7. return $T_\varepsilon$
Analysis of $\varepsilon$-Closure

- $U$ can only grow in size through each iteration
- The set $U$ cannot grow beyond the total set of states $Q$ which is finite. Hence the algorithm always terminates for any NFA $N$.
- Time complexity: $O(|Q|)$. 
Recognition using NFA

The following algorithm may be used to recognize a string using a NFA.

**Algorithm 2** Recognition using NFA

**Require:** A string \( x \in A^* \).

**Ensure:** Boolean

\[
S := \varepsilon\text{-CLOSURE}\{q_0\}.
\]

\[
a := \text{nextchar}(x)
\]

**while** \( a \neq \text{end\_of\_string} \) **do**

\[
S := \varepsilon\text{-CLOSURE}(S \xrightarrow{a})
\]

\[
a := \text{nextchar}(x)
\]

**end while**

**return** \( S \cap F \neq \emptyset \)

In the above algorithm we extend our notation for targets of transitions to include sets of sources. Thus

\[
S \xrightarrow{a} = \{q' \mid \exists q \in S : q \xrightarrow{a} q'\}
\]
Analysis of Recognition using NFA

• Even if $\varepsilon$-closure is computed as a call from within the algorithm, the time taken to recognize a string is bounded by $O(|x| \cdot |Q_{N_r}|)$ where $|Q_{N_r}|$ is the number of states in $N_r$.

• The space required for the automaton is at most $O(|r|)$.

• Given that $\varepsilon$-closure of each state can be pre-computed knowing the NFA, the recognition algorithm can run in time linear in the length of the input string $x$ i.e. $O(|x|)$.

• Knowing that the above algorithm is deterministic once $\varepsilon$-closures are pre-computed one may then work towards a Deterministic automaton to reduce the space required.
5.3. Deterministic Finite Automata (DFA)

Conversion of NFAs to DFAs
Deterministic Finite Automata

• A deterministic finite automaton (DFA) is a NFA in which
  1. there are no transitions on $\varepsilon$ and
  2. $\rightarrow$ yields a \textit{at most one} target state for each source state and symbol from $\mathcal{A}$ i.e. the transition relation is no longer a relation but a function$^a$

$$\delta : Q \times \mathcal{A} \rightarrow Q$$

• Clearly if every regular expression had a DFA which accepts the same language, all backtracking could be avoided.

$^a$Also in the case of the NFA the relation $\rightarrow$ may not define a transition from every state on every letter
Transition Tables of NFAs

We may think of a finite-state automaton as being defined by a 2-dimensional table of size $|Q| \times |A|$ in which for each state and each letter of the alphabet there is a set of possible target states defined. In the case of a non-deterministic automaton,

1. for each state there could be $\varepsilon$ transitions to
   (a) a set consisting of a single state or
   (b) a set consisting of more than one state.

2. for each state $q$ and letter $a$, there could be
   (a) an empty set of target states or
   (b) a set of target states consisting of a single state or
   (c) a set of target states consisting of more than one state
Transition Tables of DFAs

In the case of a deterministic automaton

1. there are no $\varepsilon$ transitions, and

2. for each state $q$ and letter $a$
   
   (a) either there is no transition (in which case we add a new “sink” state which is a non-accepting state)
   
   (b) or there is a transition to a unique state $q'$.

The recognition problem for the same language of strings becomes simpler and would work faster (it would have no back-tracking) if the NFA could be converted into a DFA accepting the same language.
NFA to DFA

Let $N = \langle Q_N, A \cup \{\varepsilon\}, s_N, F_N, \rightarrow_N \rangle$ be a NFA with

- $Q_N$ the set of states of the NFA
- $A$ the alphabet
- $s_N \in Q_N$ the start state of the NFA
- $F_N \subseteq Q_N$ the accepting states of the NFA and
- $\rightarrow_N \subseteq Q_N \times A \times Q_N$ the transition relation.

We would like to construct a DFA $D = \langle Q_D, A, s_D, F_D, \rightarrow_D \rangle$ where

- $Q_D$ the set of states of the DFA
- $A$ the alphabet
- $s_D \in Q_D$ the start state of the DFA
- $F_D$ the final or accepting states of the DFA and
- $\delta_D : Q_D \times A \rightarrow Q_D$ the transition function of the DFA.

We would like $\mathcal{L}(N) = \mathcal{L}(D)$.
The Subset Construction

• The $\varepsilon$-closure of each NFA state is a set of NFA states with “similar” behaviour, since they make their transitions on the same input symbols though with different numbers of $\varepsilon$s.

• Each state of the DFA refers to a *subset of states of the NFA* which exhibit “similar” behaviour. Similarity of behaviour refers to the fact that they accept the same input symbols. The behaviour of two different NFA states may not be “identical” because they may have different numbers of $\varepsilon$ transitions for the same input symbol.

• A major source of non-determinism is the presence of $\varepsilon$ transitions. The use of $\varepsilon$-CLOSEURe creates a cluster of similar states.

• Since the notion of acceptance of a string by an automaton, implies finding an *accepting sequence* even though there may be other *non-accepting sequences*, the non-accepting sequences may be ignored and those non-accepting states may be clustered with the accepting states of the NFA. So two different states reachable by the same sequence of symbols may be also thought to be similar.
Algorithm 3 Construction of DFA from NFA

Require: NFA $N = (Q_N, A \cup \{\varepsilon\}, s_N, F_N, \rightarrow_N)$

Ensure: DFA $D = (Q_D, A, s_D, F_D, \delta_D)$ with $\mathcal{L}(N) = \mathcal{L}(D)$

1. $s_D := \varepsilon$-CLOSURE($\{s_N\}$);
2. $Q_D := \{s_D\}; F_D := \emptyset; \delta_D := \emptyset$
3. $U := \{s_D\} \{U$ is the set of unvisited states of the DFA$\}$
4. while $U \neq \emptyset$ do
5. Choose any $q_D \in U; U := U - \{q_D\}$
6. for all $a \in A$ do
7. $q_D' := \varepsilon$-CLOSURE($q_D \xrightarrow{a}_N$) \{Note: $q_D \subseteq Q_N$\}
8. $\delta_D(q_D, a) := q_D'$
9. if $q_D' \cap F_N \neq \emptyset$ then
10. $F_D := F_D \cup \{q_D'\}$
11. end if
12. if $q_D' \notin Q_D$ then
13. $Q_D := Q_D \cup \{q_D'\}$
14. $U := U \cup \{q_D'\}$
15. end if
16. end for
17. end while
Example-NFA

Consider the NFA constructed for the regular expression \((a|b)^*abb\).

and apply the NFA to DFA construction algorithm
Determinising

\( N(a|b)^*abb \)

\( EC_0 = \varepsilon\text{-CLOSURE}(0) = \{0, 1, 2, 3, 7\} \)

2 \( \xrightarrow{a} \) \( N \) 4 and 7 \( \xrightarrow{a} \) \( N \) 8. So \( EC_0 \xrightarrow{a} \) \( D \) \( \varepsilon\text{-CLOSURE}(4, 8) = EC_{4,8} \). Similarly

\( EC_0 \xrightarrow{b} \) \( D \) \( \varepsilon\text{-CLOSURE}(5) = EC_5 \)

\( EC_{4,8} = \varepsilon\text{-CLOSURE}(4, 8) = \{4, 6, 7, 1, 2, 3, 8\} \)

\( EC_5 = \varepsilon\text{-CLOSURE}(5) = \{5, 6, 7, 1, 2, 3\} \)

\( EC_5 \xrightarrow{a} \) \( D \) \( \varepsilon\text{-CLOSURE}(4, 8) = EC_{4,8} \) and \( EC_5 \xrightarrow{b} \) \( D \) \( \varepsilon\text{-CLOSURE}(5) \)

\( EC_{4,8} \xrightarrow{a} \) \( D \) \( \varepsilon\text{-CLOSURE}(4, 8) = EC_{4,8} \) and \( EC_{4,8} \xrightarrow{b} \) \( D \) \( \varepsilon\text{-CLOSURE}(5) \)

\( \varepsilon\text{-CLOSURE}(5, 9) = EC_{5,9} \)

\( EC_{5,9} = \varepsilon\text{-CLOSURE}(5, 9) = \{5, 6, 7, 1, 2, 3, 9\} \)

\( EC_{5,9} \xrightarrow{a} \) \( D \) \( \varepsilon\text{-CLOSURE}(4, 8) = EC_{4,8} \) and \( EC_{5,9} \xrightarrow{b} \) \( D \) \( \varepsilon\text{-CLOSURE}(5) \)

\( \varepsilon\text{-CLOSURE}(5, 10) = EC_{5,10} \)

\( EC_{5,10} = \varepsilon\text{-CLOSURE}(5, 10) = \{5, 6, 7, 1, 2, 3, 10\} \)

\( EC_{5,10} \xrightarrow{a} \) \( D \) \( \varepsilon\text{-CLOSURE}(4, 8) \) and \( EC_{5,10} \xrightarrow{b} \) \( \varepsilon\text{-CLOSURE}(5) \)
Final DFA

\[ D_{(a|b)^*abb} \]

State Table:

<table>
<thead>
<tr>
<th>State</th>
<th>Input Symbol</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>a</td>
<td>b</td>
</tr>
<tr>
<td>( EC_0 )</td>
<td>( EC_{4,8} )</td>
<td>( EC_5 )</td>
</tr>
<tr>
<td>( EC_{4,8} )</td>
<td>( EC_{4,8} )</td>
<td>( EC_{5,9} )</td>
</tr>
<tr>
<td>( EC_5 )</td>
<td>( EC_{4,8} )</td>
<td>( EC_5 )</td>
</tr>
<tr>
<td>( EC_{5,9} )</td>
<td>( EC_{4,8} )</td>
<td>( EC_{5,10} )</td>
</tr>
<tr>
<td>( EC_{5,10} )</td>
<td>( EC_{4,8} )</td>
<td>( EC_{5,9} )</td>
</tr>
</tbody>
</table>
Scanning: 5

The Big Picture
Exercise 5.2

1. Write a regular expression to specify all numbers in binary form that are multiples of 4.

2. Write regular expressions to specify all numbers in binary form that are not multiples of 4.

3. Each comment in the C language
   - begins with the characters “//” and ends with the newline character, or
   - begins with the characters “/ *” and ends with “* /” and may run across several lines.

   (a) Write a regular expression to recognize comments in the C language.
   (b) Transform the regular expression into a NFA.
   (c) Transform the NFA into a DFA.
   (d) Explain why most programming languages do not allow nested comments.

   (e) **modified C comments.** If the character sequences “//”, “/ *” and “* /” are allowed to appear in ’quoted’ form as “’ / ’”, “’ / *’” and “’ * / ’” respectively within a C comment, then give
      i. a modified regular expression for C comments
      ii. a NFA for these modified C comments
      iii. a corresponding DFA for modified C comments

4. Many systems such as Windows XP and Linux recognize commands, filenames and folder names by the their shortest unique prefix. Hence given the 3 commands `chmod`, `chgrp` and `chown`, their shortest unique prefixes are respectively `chm`, `chg` and `cho`. A user can type the shortest unique prefix of the command and the system will automatically complete it for him/her.
(a) Draw a DFA which recognizes all prefixes that are at least as long as the shortest unique prefix of each of
the above commands.

(b) Suppose the set of commands also includes two more commands **cmp** and **cmpdir**, state how you will
include such commands also in your DFA where one command is a prefix of another.
6. Parsing or Syntax Analysis

6.1. Grammars

Parsing Or Syntax Analysis
Formal languages: Definition, Recognition, Generation

There are three different processes used in dealing with a formal language.

**Definition**: Regular expressions is a formal (functional programming) language used to define or specify a formal language of tokens.

**Recognition**: Automata are the standard mechanism used to recognize words/phrases of a formal language. An automaton is used to determine whether a given word/phrase is a member of the formal language defined in some other way.

**Generation**: Grammars are used to define the generation of the words/phrases of a formal language.
Non-regular language

Consider the following two languages over an alphabet $A = \{a, b\}$.

$$R = \{a^nb^n|n < 100\}$$
$$P = \{a^nb^n|n > 0\}$$

• $R$ may be finitely represented by a regular expression (even though the actual expression is very long).
• However, $P$ cannot actually be represented by a regular expression.
• A regular expression is not powerful enough to represent languages which require parenthesis matching to arbitrary depths.
• All high level programming languages require an underlying language of expressions which require parentheses to be nested and matched to arbitrary depth.
6.2. Context-Free Grammars

Grammars

Definition 6.1 A grammar \( G = \langle N, T, P, S \rangle \) consists of

- a set \( N \) of nonterminal symbols, or variables,
- a start symbol \( S \in N \),
- a set \( T \) of terminal symbols or the alphabet,
- a set \( P \) of productions or rewrite rules where each rule is of the form \( \alpha \to \beta \) for \( \alpha, \beta \in (N \cup T)^* \)

Definition 6.2 Given a grammar \( G = \langle N, T, P, S \rangle \), any \( \alpha \in (N \cup T)^* \) is called a sentential form. Any \( x \in T^* \) is called a sentence\(^a\).

Note. Every sentence is also a sentential form.

\(^a\)some authors call it a word. However we will reserve the term word to denote the tokens of a programming language.
Grammars: Notation

- Upper case roman letters ($A, B, \ldots, X, Y,$ etc.) denote nonterminals.
- Final upper case roman letters ($X, Y, Z$ etc.) may also be used as meta-variables which denote arbitrary non-terminal symbols of a grammar.
- Initial lower case roman letters ($a, b, c$ etc.) will be used to denote terminal symbols.
- Lower case greek letters ($\alpha, \beta$ etc.) denote sentential forms (or even sentences).
- Final lower case letters ($u, v, \ldots, x, y, z$ etc.) denote only sentences.
- In each case the symbols could also be decorated with sub-scripts or super-scripts.
Context-Free Grammars: Definition

Definition 6.3 A grammar $G = \langle N, T, P, S \rangle$ is called context-free if each production is of the form $X \rightarrow \alpha$, where

- $X \in N$ is a nonterminal and
- $\alpha \in (N \cup T)^*$ is a sentential form.
- The production is terminal if $\alpha$ is a sentence
CFG: Example 1

\[ G = \langle \{S\}, \{a, b\}, P, S \rangle, \text{ where } S \rightarrow ab \text{ and } S \rightarrow aSb \text{ are the only productions in } P. \]

Derivations look like this:

- \[ S \Rightarrow ab \]

- \[ S \Rightarrow aSb \Rightarrow aabb \]

- \[ S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaabbb \]

- \[ S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aaaSbbb \]

The first three derivations are complete while the last one is partial.
Derivations

Definition 6.4 A (partial) derivation (of length \( n \in \mathbb{N} \)) in a context-free grammar is a finite sequence of the form

\[
\alpha_0 \Rightarrow \alpha_1 \Rightarrow \alpha_2 \Rightarrow \cdots \alpha_n
\]  

(3)

where each \( \alpha_i \in (N \cup T)^* \) (\( 0 \leq i \leq n \)) is a sentential form where \( \alpha_0 = S \) and \( \alpha_{i+1} \) is obtained by applying a production rule to a non-terminal symbol in \( \alpha_i \) for \( 0 \leq i < n \).

Notation. \( S \Rightarrow^* \alpha \) denotes that there exists a derivation of \( \alpha \) from \( S \).

Definition 6.5 The derivation (3) is complete if \( \alpha_n \in T^* \) i.e. \( \alpha_n \) is a sentence. Then \( \alpha_n \) is said to be a sentence generated by the grammar.
Language Generation

Definition 6.6 The language generated by a grammar $G$ is the set of sentences that can be generated by $G$ and is denoted $L(G)$.

Example 6.7 $L(G)$, the language generated by the grammar $G$ is $\{a^n b^n | n > 0\}$. Prove using induction on the length of derivations.
Regular Grammars

Definition 6.8 A production rule of a context-free grammar is Right Linear: if it is of the form $X \rightarrow a$ or $X \rightarrow aY$

Left Linear: if it is of the form $X \rightarrow a$ or $X \rightarrow Y a$

where $a \in T$ and $X, Y \in N$.

Definition 6.9 A regular grammar is a context-free grammar whose productions are either only right linear or only left linear.
Consider the DFA with the states renamed as shown above. We could easily convert the DFA to a right linear grammar which generates the language accepted by the DFA.
CFG: Empty word

\[ G = \langle \{S\}, \{a,b\}, P, S \rangle, \text{ where } S \rightarrow SS \mid aSb \mid \varepsilon \]
generates all sequences of matching nested parentheses, including the empty word \( \varepsilon \).

A leftmost derivation might look like this:

\[
S \Rightarrow SS \Rightarrow SSS \Rightarrow SS \Rightarrow aSbS \Rightarrow abS \Rightarrow abaSb \ldots
\]

A rightmost derivation might look like this:

\[
S \Rightarrow SS \Rightarrow SSS \Rightarrow SS \Rightarrow SaSb \Rightarrow Sab \Rightarrow aSbab \ldots
\]

Other derivations might look like \textit{God alone knows what!}

\[
S \Rightarrow SS \Rightarrow SSS \Rightarrow SS \Rightarrow \ldots
\]

Could be quite confusing!
CFG: Derivation trees 1

Derivation sequences

• put an artificial order in which productions are fired.

• instead look at trees of derivations in which we may think of productions as being fired in parallel.

• There is then no highlighting in red to determine which copy of a non-terminal was used to get the next member of the sequence.

• Of course, generation of the empty word $\varepsilon$ must be shown explicitly in the tree.
CFG: Derivation trees 2

Derivation tree of

\[ abaabb \]
Another Derivation tree of \textit{abaabb}
Yet another Derivation tree of $abaabb$
6.3. Ambiguity

Ambiguity Disambiguation
Ambiguity: 1

\[ G_1 = \langle \{ E, I, C \}, \{ y, z, 4, *, + \}, P_1, \{ E \} \rangle \]

where \( P_1 \) consists of the following productions.

\[
\begin{align*}
E & \rightarrow I \mid C \mid E + E \mid E \ast E \\
I & \rightarrow y \mid z \\
C & \rightarrow 4
\end{align*}
\]

Consider the sentence \( y + 4 \ast z \).
Ambiguity: 2

\[ G_1 = \langle \{ E, I, C \}, \{ y, z, 4, *, + \}, P_1, \{ E \} \rangle \] where \( P_1 \) consists of the following productions.

\[
\begin{align*}
E & \rightarrow I \mid C \mid E+E \mid E*E \\
I & \rightarrow y \mid z \\
C & \rightarrow 4
\end{align*}
\]

Consider the sentence \( y + 4 * z \).
Ambiguity: 3

\[ G_1 = \langle \{ E, I, C \}, \{ y, z, 4, *, + \}, P_1, \{ E \} \rangle \] where \( P_1 \) consists of the following productions.

\[
\begin{align*}
E & \rightarrow I \mid C \mid E + E \mid E \ast E \\
I & \rightarrow y \mid z \\
C & \rightarrow 4
\end{align*}
\]

Consider the sentence \( y + 4 \ast z \).
Ambiguity: 4

\[ G_1 = \langle \{E, I, C\}, \{y, z, 4, *, +\}, P_1, \{E\}\rangle \]

where \( P_1 \) consists of the following productions.

\[
E \rightarrow I \mid C \mid E + E \mid E * E \\
I \rightarrow y \mid z \\
C \rightarrow 4
\]

Consider the sentence \( y + 4 * z \).
Ambiguity: 5

\[ G_1 = \langle \{ E, I, C \}, \{ y, z, 4, *, + \}, P_1, \{ E \} \rangle \] where \( P_1 \) consists of the following productions.

\[
\begin{align*}
E & \rightarrow I \mid C \mid E + E \mid E * E \\
I & \rightarrow y \mid z \\
C & \rightarrow 4
\end{align*}
\]

Consider the sentence \( y + 4 * z \).
Left-most Derivation 1

Left-most derivation of $y+4*z$ corresponding to the first derivation tree.

$$
\begin{align*}
E & \Rightarrow E + I \\
E + E & \Rightarrow I + C \\
y + I & \Rightarrow y + C \\
y + * I & \Rightarrow y + 4 * E \\
y + 4 * I & \Rightarrow y + 4 * z
\end{align*}
$$
Left-most Derivation 2

Left-most derivation of $y+4*z$ corresponding to the *second* derivation tree.

\[
\begin{align*}
E & \Rightarrow \\
E*E & \Rightarrow \\
E+E*E & \Rightarrow \\
I+E*E & \Rightarrow \\
y+E*E & \Rightarrow \\
y+C*E & \Rightarrow \\
y + 4*E & \Rightarrow \\
y + 4*I & \Rightarrow \\
y + 4 * z & \Rightarrow 
\end{align*}
\]
Right-most Derivation 1

Right-most derivation of $y+4*z$ corresponding to the *first* derivation tree.

$$
\begin{align*}
E & \Rightarrow \\
E + E & \Rightarrow \\
E + E * E & \Rightarrow \\
E + E * I & \Rightarrow \\
E + E * z & \Rightarrow \\
E + C * z & \Rightarrow \\
E + 4 * z & \Rightarrow \\
I + 4 * z & \Rightarrow \\
y + 4 * z & 
\end{align*}
$$
Right-most Derivation 2

Right-most derivation of $y+4*z$ corresponding to the second derivation tree.

$$
E 
\Rightarrow
E*E 
\Rightarrow
E*I 
\Rightarrow
E*z 
\Rightarrow
E+E*z 
\Rightarrow
E+C*z 
\Rightarrow
E+4*z 
\Rightarrow
I+4*z 
\Rightarrow
y + 4*z
$$
Characterizing Ambiguity

The following statements are equivalent.

• A CFG is ambiguous if some sentence it generates has more than one derivation tree

• A CFG is ambiguous if there is a some sentence it generates with more than one left-most derivation

• A CFG is ambiguous if there is a some sentence it generates with more than one right-most derivation
Disambiguation

The only way to remove ambiguity (without changing the language generated) is to change the grammar by introducing some more non-terminal symbols and changing the production rules. Consider the grammar $G'_1 = \langle N', \{y, z, 4, *, +\}, P', \{E\} \rangle$ where $N' = N \cup \{T, F\}$ with the following production rules $P'$.

$$
\begin{align*}
E & \rightarrow E + T \mid T \\
T & \rightarrow T \ast F \mid F \\
F & \rightarrow I \mid C \\
I & \rightarrow y \mid z \\
C & \rightarrow 4
\end{align*}
$$

and compare it with the grammar $G_1$
Left-most Derivation 1’

The left-most derivation of $y+4*z$ is then as follows.

$$
\begin{align*}
E & \Rightarrow \\
E+T & \Rightarrow \\
I+T & \Rightarrow \\
y+T & \Rightarrow \\
y+T*F & \Rightarrow \\
y+T*F & \Rightarrow \\
y+F*F & \Rightarrow \\
y+C*F & \Rightarrow \\
y+4*F & \Rightarrow \\
y+4*I & \Rightarrow \\
y + 4 * z
\end{align*}
$$
Left-most Derivations

Compare it with the Left-most Derivation 1.

\[ G_1. \quad E \Rightarrow E+E \Rightarrow I+E \Rightarrow y+E \Rightarrow y+E*E \Rightarrow \]
\[ \quad y+C*E \Rightarrow y+4*E \Rightarrow y+4*I \Rightarrow y + 4 * z \]

\[ G'_1. \quad E \Rightarrow E+T \Rightarrow I+T \Rightarrow y+T \Rightarrow y+T*F \Rightarrow y+T*F \Rightarrow y+F*F \Rightarrow \]
\[ \quad y+C*F \Rightarrow y+4*F \Rightarrow y+4*I \Rightarrow y + 4 * z \]

There is no derivation in \( G'_1 \) corresponding to Left-most Derivation 2 (Why not?).
Right-most Derivation 1’

Right-most derivation of $y+4*z$ corresponding to the *first* derivation tree.

$$
E \Rightarrow \\
E+T \Rightarrow \\
E+T*F \Rightarrow \\
E+T*I \Rightarrow \\
E+T*z \Rightarrow \\
E+C*z \Rightarrow \\
E+4*z \Rightarrow \\
F+4*z \Rightarrow \\
I+4*z \Rightarrow \\
+4*z \Rightarrow \\
y+4*z
$$

Compare it with the Right-most Derivation 1.

There is no derivation corresponding to Right-most Derivation 2.
Disambiguation by Parenthesization

Another method of disambiguating a language is to change the language generated, by introducing suitable bracketing mechanisms.

Example 6.10 Compare the following fully parenthesized grammar $G_2$ (which has the extra terminal symbols ( and )) with the grammar $G_1$ without parentheses

\[
E \rightarrow I \mid C \mid (E+E) \mid (E\ast E) \\
I \rightarrow y \mid z \\
C \rightarrow 4
\]

Though unambiguous, the language defined by this grammar is different from that of the original grammar without parentheses.
Associativity and Precedence

The grammar $G'_1$ implements

**Precedence.** $\ast$ has higher precedence than $\mp$.

**Associativity.** $\ast$ and $\mp$ are both left associative operators.
Exercise 6.1

1. Two context-free grammars are considered equivalent if they generate the same language. Prove that $G_1$ and $G'_1$ are equivalent.

2. Palindromes. A palindrome is a string that is equal to its reverse i.e. it is the same when read backwards (e.g. $aabbaa$ and $abaabaaba$ are both palindromes). Design a grammar for generating all palindromes over the terminal symbols $a$ and $b$.

3. Matching brackets.
   
   (a) Design a context-free grammar to generate sequences of matching brackets when the set of terminals consists of three pairs of brackets $\{ (, ), [ , ], \{ , \} \}$.
   
   (b) If your grammar is ambiguous give two rightmost derivations of the same string and draw the two derivation trees. Explain how you will modify the grammar to make it unambiguous.
   
   (c) If your grammar is not ambiguous prove that it is not ambiguous.

4. Design an unambiguous grammar for the expression language on integers consisting of expressions made up of operators $+,-,\ast,/,%$ and the bracketing symbols $(,)$, assuming the usual rules of precedence among operators that you have learned in school.

5. Modify the above grammar to include the exponentiation operator $^\wedge$ which has a higher precedence than the other operators and is right-associative.

6. How will you modify the grammar above to include the unary minus operator $-$ where the unary minus has a higher precedence than other operators?

7. The language specified by a regular expression can also be generated by a context-free grammar.
(a) Design a context-free grammar to generate all floating-point numbers allowed by the C language.
(b) Design a context-free grammar to generate all numbers in binary form that are not multiples of 4.
(c) Write a regular expression to specify all numbers in binary form that are multiples of 3.

8. Prove that the $G'_1$ is indeed unambiguous.

9. Prove that the grammar of fully parenthesized expressions is unambiguous.

10. Explain how the grammar $G'_1$ implements left associativity and precedence.
Introduction to Parsing
Overview of Parsing

Since

• parsing requires the checking whether a given token stream conforms to the rules of the grammar and

• since a context-free grammar may generate an infinite number of different strings

any parsing method should be guided by the given input (token) string, so that a deterministic strategy may be evolved.
Parsing Methods

Two kinds of parsing methods

**Top-down parsing** Try to *generate* the given input sentence from the start symbol of the grammar by applying the production rules.

**Bottom-up parsing** Try to *reduce* the given input sentence to the start symbol by applying the rules in *reverse*

In general top-down parsing requires long *look-aheads* in order to do a deterministic guess from the given input token stream. On the other hand bottom-up parsing yields better results and can be automated by software tools.
Reverse of Right-most Derivations

The result of a Bottom-Up Parsing technique is usually to produce a reverse of the right-most derivation of a sentence.

Example For the ambiguous grammar $G_1$ and corresponding to the right-most derivation 2 we get

\[
\begin{align*}
y + 4 \times z & \leftarrow \\
I + 4 \times z & \leftarrow \\
E + 4 \times z & \leftarrow \\
E + C \times z & \leftarrow \\
E + E \times z & \leftarrow \\
E \times z & \leftarrow \\
E \times I & \leftarrow \\
E \times E & \leftarrow \\
E & \leftarrow
\end{align*}
\]
Bottom-Up Parsing Strategy

The main problem is to match parentheses of arbitrary nesting depths. This requires a stack data structure to do the parsing so that unbounded nested parentheses and varieties of brackets may be matched. Our basic parsing strategy is going to be based on a technique called *shift-reduce* parsing.

**shift.** Refers to moving the next token from the input token stream into a *parsing* stack.

**reduce.** Refers to applying a production rule in reverse i.e. given a production $X \rightarrow \alpha$ we reduce any occurrence of $\alpha$ in the parsing stack to $X$. 
Fully Bracketed Expression

Consider an example of a fully bracketed expression
Parsing: FB0

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

\[
\begin{align*}
r_1 &: E \rightarrow E \rightarrow T \\
r_2 &: E \rightarrow T \\
r_3 &: T \rightarrow T / D \\
r_4 &: T \rightarrow D \\
r_5 &: D \rightarrow a | b | (E) \\
\end{align*}
\]
Parsing: FB1

Principle:
Reduce whenever possible.
Shift only when reduce is impossible.
Parsing: FB2

r1. $E \rightarrow E \cdot T$
r2. $E \rightarrow T$
r3. $T \rightarrow T \cdot D$
r4. $T \rightarrow D$
r5. $D \rightarrow \text{a} \mid \text{b} \mid (E)$

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

$\rightarrow (\text{a} \cdot \text{b} \cdot )$
Parsing: FB3

Principle:
Reduce whenever possible.
Shift only when reduce is impossible.

Reduce
## Parsing: FB4

**Principle:**
Reduce whenever possible. Shift only when reduce is impossible.
Parsing: FB5

Principle:
Reduce whenever possible. Shift only when reduce is impossible.
Parsing: FB6

r1. E → E T
r2. E → T
r3. T → T ( D
r4. T → D
r5. D → a | b | ( E )

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

E
( Shift
Parsing: FB7

Principle:
Reduce whenever possible. 
Shift only when reduce is impossible
Parsing: FB8

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

- E → E + T
- E → T
- T → T / D
- T → D
- D → a | b | (E)

Reduce
Parsing: FB9

r1. E → E T
r2. E → T
r3. T → T D
r4. T → D
r5. D → a | b | ( E )

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

Reduce

D

( )

E

−
Parsing: FB10

Principle:
Reduce whenever possible.
Shift only when reduce is impossible
Principle:
Reduce whenever possible.
Shift only when reduce is impossible
Parsing: FB12

r1. E → E T
r2. E → T
r3. T → T T D
r4. T → D
r5. D → a | b | ( ( E ) )

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

Shift
Parsing: FB13

Principle:
Reduce whenever possible.
Shift only when reduce is impossible.
Parsing: FB14

r1. $E \rightarrow E \cdot T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \cdot D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

Principle:
Reduce whenever possible.
Shift only when reduce is impossible.

Reduce
Parsing: FB15

Principle:
Reduce whenever possible. Shift only when reduce is impossible

Reduce?
Parsing: FB16

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

No, REDUCE!
Parsing: FB17

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

Reduce?
Parsing: FB18

Principle:
Reduce whenever possible. Shift only when reduce is impossible.
Parsing: FB19

Principle:
Reduce whenever possible. Shift only when reduce is impossible.

- r1. \( E \rightarrow E \rightarrow T \)
- r2 \( E \rightarrow T \)
- r3 \( T \rightarrow T \rightarrow D \)
- r4 \( T \rightarrow D \)
- r5 \( D \rightarrow a \mid b \mid (E) \)

Reduce
Principle:
Reduce whenever possible.
Shift only when reduce is impossible

r1. E → E T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | (E )
Parsing: FB21

Principle:
Reduce whenever possible.
Shift only when reduce is impossible.

r1. $E \rightarrow E \cdot T$

r2 $E \rightarrow T$

r3 $T \rightarrow T \cdot D$

r4 $T \rightarrow D$

r5 $D \rightarrow a \mid b \mid (E)$

Reduce?
Parsing: FB22

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

No, REDUCE!
Parsing: FB23

Principle:
Reduce whenever possible.
Shift only when reduce is impossible
Parsing: FB24

**Principle:**
Reduce whenever possible. Shift only when reduce is impossible.
Principle:
Reduce whenever possible.
Shift only when reduce is impossible
Parsing: FB26

Principle:
Reduce whenever possible.
Shift only when reduce is impossible

```plaintext
r1. E → E T
r2. E → T
r3. T → T D
r4. T → D
r5. D → a | b | ( E )
```

Reduce
Parsing: FB27

Principle:
Reduce whenever possible. Shift only when reduce is impossible

r1. E → E T
r2 E → T
r3 T → T D
r4 T → D
r5 D → a | b | ( E )

Reduce
Unbracketed Expression

Consider an example of an unbracketed expression which relies on the precedence rules as defined in the grammar.
Parsing: UB0

r1. E → E + T
r2. E → T
r3. T → T | D
r4. T → D
r5. D → a | b | (E)

a ← a / b
Parsing: UB1

Principle:
Reduce whenever possible.
Shift only when reduce is impossible.

Shift

r1. \( E \rightarrow E \cdot T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \cdot D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \mid b \mid ( \cdot E ) \)
Parsing: UB2

r1. $E \rightarrow E \varepsilon T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \, / \, D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \, | \, b \, | \, ( \, E \, )$

Reduce by r5
Parsing: UB3

r1. \( E \rightarrow E \cdot T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \cdot D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \mid b \mid (E) \)

Reduce by r4
Parsing: UB4

r1. E → E T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | ( E )

Reduce by r2
Parsing: UB5

r1. $E \rightarrow E \mathbin{-} T$

r2. $E \rightarrow T$

r3. $T \rightarrow T / D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

\[\text{Shift}\]

\[E\]
Parsing: UB6

r1. $E \rightarrow E \cdot T$
r2. $E \rightarrow T$
r3. $T \rightarrow T \cdot D$
r4. $T \rightarrow D$
r5. $D \rightarrow a \mid b \mid (E)$
Parsing: UB7

r1. $E \rightarrow E T$

r2. $E \rightarrow T$

r3. $T \rightarrow T D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

Reduce by r5
Parsing: UB8

r1. \( E \rightarrow E \rightarrow T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \rightarrow D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \mid b \mid (E) \)

Reduce by r4
Parsing: UB8a

r1. E → E T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | ( E )

Reduce by r4
Parsing: UB9a

\[ r1. \ E \rightarrow E \rightarrow T \]
\[ r2. \ E \rightarrow T \]
\[ r3. \ T \rightarrow T \rightarrow D \]
\[ r4. \ T \rightarrow D \]
\[ r5. \ D \rightarrow a \mid b \mid (E) \]

Reduce by r1
Parsing: UB10a

r1. $E \rightarrow E \rightarrow T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \rightarrow D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$
Parsing: UB11a

r1. E → E + T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | ( E )

Shift
Parsing: UB12a

\[ 
\begin{align*}
  r1. & \quad E \rightarrow E \rightarrow T \\
  r2. & \quad E \rightarrow T \\
  r3. & \quad T \rightarrow T \rightarrow D \\
  r4. & \quad T \rightarrow D \\
  r5. & \quad D \rightarrow a \mid b \mid (E) \\
\end{align*} 
\]

Reduce by r5
Parsing: UB13a

r1. $E \rightarrow E \cdot T$

r2. $E \rightarrow T$

r3. $T \rightarrow T / D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

Reduce by r4
Parsing: UB14a

r1. \( E \rightarrow E \oplus T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \oplus D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \mid b \mid (E) \)

Stuck!

Get back!

Reduce by r2
Parsing: UB14b

r1. E → E T
r2. E → T
r3. T → T + D
r4. T → D
r5. D → a | b | (E)

Get back!

Reduce by r2
Parsing: UB13b

r1. \( E \rightarrow E \rightarrow T \)

r2. \( E \rightarrow T \)

r3. \( T \rightarrow T \rightarrow D \)

r4. \( T \rightarrow D \)

r5. \( D \rightarrow \text{a} \mid \text{b} \mid (\text{E}) \)

Get back!

Reduce by r4
Parsing: UB12b

r1. E → E T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | (E)

Get back!

Reduce by r5
Parsing: UB11b

r1. \( E \rightarrow E \leftarrow T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T /\backslash D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \) | \( b \) | \( ( E ) \)

Get back!  Shift
Parsing: UB10b

r1. \( E \rightarrow E \rightarrow T \)

r2. \( E \rightarrow T \)

r3. \( T \rightarrow T \rightarrow D \)

r4. \( T \rightarrow D \)

r5. \( D \rightarrow a \mid b \mid ( E ) \)

Get back!

Shift
Parsing: UB9b

r1. \( E \rightarrow E \cdot T \)

r2 \( E \rightarrow T \)

r3 \( T \rightarrow T \cdot D \)

r4 \( T \rightarrow D \)

r5 \( D \rightarrow a \mid b \mid ( E ) \)

Get back to where you once belonged!

Reduce by r1
Parsing: UB8b

Principle:
Reduce whenever possible, but depending upon lookahead.

Shift instead of reduce here!

Shift–reduce conflict

Reduce by r4
Parsing: UB8

r1. \( E \rightarrow E \rightarrow T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \rightarrow D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \mid b \mid (E) \)

Reduce by r4
Parsing: UB9

r1. E → E + T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | (E)

Shift
Parsing: UB10

r1. E → E T
r2. E → T
r3. T → T D
r4. T → D
r5. D → a | b | ( E )
Parsing: UB11

r1. $E \rightarrow E \cdot T$

r2 $E \rightarrow T$

r3 $T \rightarrow T \cdot D$

r4 $T \rightarrow D$

r5 $D \rightarrow a \mid b \mid (E)$

Reduce by r5
Parsing: UB12

r1. E → E T
r2. E → T
r3. T → T D
r4. T → D
r5. D → a | b | (E )

Reduce by r3
Parsing: UB13

r1. $E \rightarrow E \cdot T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \cdot D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

Reduce by r1
6.5. Bottom-Up Parsing

Bottom-Up Parsing
Parse Trees: 0

r1. \[ E \rightarrow E \rightarrow T \]

r2. \[ E \rightarrow T \]

r3. \[ T \rightarrow T \rightarrow D \]

r4. \[ T \rightarrow D \]

r5. \[ D \rightarrow a | b | (E) \]
Parse Trees: 1

r1. E → E − T
r2. E → T
r3. T → T / D
r4. T → D
r5. D → a | b | ( E )

D
a
−
a
/
b
Parse Trees: 2

r1. \( E \rightarrow E \rightarrow^* T \)

r2. \( E \rightarrow T \)

r3. \( T \rightarrow T \rightarrow^* D \)

r4. \( T \rightarrow D \)

r5. \( D \rightarrow a \mid b \mid (E) \)

Diagram:

- \( T \)
- \( D \)
- \( a \)
- \( - \)
- \( a \)
- \( / \)
- \( b \)

Shift-reduce parsing: 2
Parse Trees: 3

r1. E → E ▶ T
r2. E → T
r3. T → T / ▶ D
r4. T → D
r5. D → a | b | (E)
Parse Trees: 3a

r1. \( E \rightarrow E \rightarrow T \)

r2. \( E \rightarrow T \)

r3. \( T \rightarrow T \rightarrow D \)

r4. \( T \rightarrow D \)

r5. \( D \rightarrow a \mid b \mid (E) \)

shift-reduce parsing
Parse Trees: 3b

r1. $E \rightarrow E T$

r2. $E \rightarrow T$

r3. $T \rightarrow T D$

r4. $T \rightarrow D$

r5. $D \rightarrow a | b | (E)$
Parse Trees: 4

r1. $E \rightarrow E \rightarrow T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \rightarrow D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

shift-reduce parsing
Parse Trees: 5

r1. $E \rightarrow E \, T$

r2 $E \rightarrow T$

r3 $T \rightarrow T \mathbin{\langle \rangle} D$

r4 $T \rightarrow D$

r5 $D \rightarrow a \mid b \mid (E)\rangle$

shift-reduce parsing
Parse Trees: 5a

r1. $E \rightarrow E \rightarrow T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \rightarrow D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

shift-reduce parsing
Parse Trees: 5b

r1. $E \rightarrow E \; T$

r2. $E \rightarrow T$

r3. $T \rightarrow T \; D$

r4. $T \rightarrow D$

r5. $D \rightarrow a \mid b \mid (E)$

shift-reduce parsing
Parse Trees: 6

r1. \( E \rightarrow E \rightarrow T \)

r2. \( E \rightarrow T \)

r3. \( T \rightarrow T \rightarrow D \)

r4. \( T \rightarrow D \)

r5. \( D \rightarrow a \mid b \mid (E) \)

shift-reduce parsing
Parse Trees: 7

r1. \( E \rightarrow E \rightarrow T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \rightarrow D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \mid b \mid (E) \)

- shift-reduce parsing
Parse Trees: 8

r1. \( E \rightarrow E \quad T \)
r2. \( E \rightarrow T \)
r3. \( T \rightarrow T \quad D \)
r4. \( T \rightarrow D \)
r5. \( D \rightarrow a \quad b \quad (E) \)

**Shift-reduce parsing**
Parsing: Summary: 1

- All high-level languages are designed so that they may be parsed in this fashion with only a single token look-ahead.
- Parsers for a language can be automatically constructed by parser-generators such as Yacc, Bison, ML-Yacc and CUP in the case of Java.
- Shift-reduce conflicts if any, are automatically detected and reported by the parser-generator.
- Shift-reduce conflicts may be avoided by suitably redesigning the context-free grammar.
Parsing: Summary: 2

• Very often shift-reduce conflicts may occur because of the prefix problem. In such cases many parser-generators resolve the conflict in favour of shifting.

• There is also a possibility of reduce-reduce conflicts. This usually happens when there is more than one nonterminal symbol to which the contents of the stack may reduce.

• A minor reworking of the grammar to avoid redundant non-terminal symbols will get rid of reduce-reduce conflicts.

The Big Picture
6.6. Simple LR Parsing

Parsing Problems 1

The main question in shift-reduce parsing is:

*When to shift and when to reduce?*

To answer this question we require

- more information from the input token stream,
- to look at the rest of the input token stream and then take a decision.

But the decision has to be automatic. So the parser requires some rules. Once given the rules we may construct the parser to follow the rules.
Parsing Problems 2

But for a very large program it may be impossible to look at *all* the input before taking a decision. So clearly the parser can look at only a limited amount of the input to take a decision. So

The next question:

How much of the input token stream would the parser require?

Disregarding the very next input token as always available, the length of the extra amount of input required for a shift-reduce decision is called the lookahead.
Parsing Problems 3

Once all the input has been read, the parser should be able to decide

in case of a valid sentence that it should only apply reduction rules and at-
tempt to reach the start symbol of the grammar only through reduc-
tions and

in case of an invalid sentence that a grammatical error has occurred in the
parsing process

To solve this problem we augment every grammar with a new start sym-
bol $S$ and a new terminal token $\$ and augment the grammar with a new
special rule. For our previous grammar we have the new rule

$$S \rightarrow E\$$
Augmented Grammar

Consider the following (simplified) augmented grammar with a single binary operator — and parenthesis. We also number the rules.

1. $S \rightarrow E\$
2. $E \rightarrow E - T$
3. $E \rightarrow T$
4. $T \rightarrow a$
5. $T \rightarrow (E)$
LR(0) Languages

LR(0) languages are those context-free languages that may be parsed by taking *deterministic shift-reduce decisions* only based on the contents of the parsing stack and without viewing any lookahead.

- “L” refers to reading the input from *left to right*,
- “R” refers to the *(reverse) of rightmost derivation*
- “0” refers to *no-lookahead*.

Many simple CFLs are LR(0). But the LR(0) parsing method is too weak for most high-level programming languages.

But understanding the LR(0) parsing method is most crucial for understanding other more powerful LR-parsing methods which require lookaheads for deterministic *shift-reduce* decision-making.
LR-Parsing Invariant

In any LR-parsing technique the following invariant holds.

For any syntactically valid sentence generated by the augmented grammar, the concatenation of the stack contents with the rest of the input gives a sentential form of a rightmost derivation.

Hence given at any stage of the parsing if \( \alpha \in (N \cup T)^* \) is the contents of the parsing stack and \( x \in T^*\$ \) is the rest of the input that has not yet been read, then \( \alpha x \) is a sentential form of a right-most derivation.
LR(0) Item

An LR(0) item consists of an LR(0) production rule with a special marker ▲ on the right hand side of rule.

• The marker is different from any of the terminal or nonterminal symbols of the grammar.

• The marker separates the contents of the stack from the expected form of some prefix of the rest of the input.

• Given a rule $X \rightarrow \alpha$, where $X$ is a nonterminal symbol and $\alpha$ is a string consisting of terminal and non-terminal symbols, an LR(0) item is of the form

\[
X \rightarrow \beta \▲ \gamma
\]

where $\alpha = \beta \gamma$.

• For each rule $X \rightarrow \alpha$, there are $|\alpha| + 1$ distinct LR(0) items – one for each position in $\alpha$.  

What does an LR(0) item signify?

The LR(0) item

\[ X \rightarrow \beta \uparrow \gamma \]

signifies that at some stage of parsing

• \( \beta \) is the string (of terminals and nonterminals) on the top of the stack and

• some prefix of the rest of the input can be generated by \( \gamma \) so that whenever \( \beta \gamma \) appears on the stack, \( \beta \gamma \) may be reduced immediately to \( X \).
LR0 Parsing Strategy

The LR0 parsing strategy is to
1. construct a DFA whose alphabet is $N \cup T \cup \{\$\}$
2. use the parsing stack to perform reductions at appropriate points

The LR0 parsing table is hence a DFA with 3 kinds of entries.

- **shift** $i$ in which a terminal symbol is shifted on to the parsing stack and the DFA moves to state $i$.

- **reduce** $j$ a reduction using the production rule $j$ is performed

- **goto** $k$ Based on the contents of the stack, the DFA moves to state $k$. 
Favourite Example

Consider our favourite augmented grammar

1. $S \rightarrow E$
2. $E \rightarrow E - T$
3. $E \rightarrow T$
4. $T \rightarrow a$
5. $T \rightarrow (E)$
Rule 1: Items

Rule 1

\[ R1. \quad S \rightarrow E\$ \]

has the following three items

\[ I1.1 \quad S \rightarrow \blacktriangle E\$ \]
\[ I1.2 \quad S \rightarrow E \blacktriangle \$ \]
\[ I1.3 \quad S \rightarrow E \$ \blacktriangle \]

one for each position on the right hand side of the rule.
Rule 2: Items

Rule 2

\[ R2. \ E \rightarrow \ E \rightarrow T \]

has the following items

\[ I2.1 \ E \rightarrow \ E \rightarrow T \]
\[ I2.2 \ E \rightarrow \ E \rightarrow T \]
\[ I2.3 \ E \rightarrow \ E \rightarrow T \]
\[ I2.4 \ E \rightarrow \ E \rightarrow T \]
Rule 3: Items

Rule 3

\[ R3. \ E \rightarrow T \]

has just the items

\[ I3.1 \ E \rightarrow \uparrow T \]
\[ I3.2 \ E \rightarrow T \uparrow \]
Rule 4: Items

Rule 4 has the items

\[ R4. \ T \rightarrow a \]

\[ I4.1 \ T \rightarrow \uparrow a \]
\[ I4.2 \ T \rightarrow a \uparrow \]
Rule 5: Items

Rule 5

\[ R5. \ T \rightarrow (E) \]

has the items

\[ I5.1 \ T \rightarrow \uparrow(E) \]
\[ I5.2 \ T \rightarrow (\uparrow E) \]
\[ I5.3 \ T \rightarrow (E\uparrow) \]
\[ I5.4 \ T \rightarrow (E)\uparrow \]
Significance of $I1.$

$I1.1 \; S \rightarrow \uparrow E\$. Hence

1. The parsing stack is empty and
2. the entire input (which has not been read yet) should be reducible to $E$ followed by the $\$. 

$I1.2 \; S \rightarrow E\uparrow\$. Hence

1. $E$ is the only symbol on the parsing stack and
2. the rest of the input consists of the terminating symbol $\$. 

$I1.3 \; S \rightarrow E\$\uparrow$. Hence

1. There is no input left to be read and
2. the stack contents may be reduced to the start symbol
DFA States: Initial and Final

• Clearly the *initial* state $S_1$ of the DFA will correspond to item I1.1.
• There should be a state corresponding to item I1.2.
• There should be a *goto* transition on the nonterminal symbol $E$ from the *initial state* (corresponding to item I1.1) to the state corresponding to item I1.2.
• The *accepting* state of the DFA will correspond to item I1.3.
• There would also be a *shift* transition on $\$$ from the state corresponding to item I1.2 to the accepting state corresponding to item I1.3.
• There should be a *reduce* action using rule 1 when the DFA reaches the state corresponding to item I1.3.
Input Possibilities

Consider item I1.1.

1. How will a grammatically valid sentence input reduce to $E\$? From the grammar it is obvious that this can happen only if the input is of a form such that

   (a) it can be reduced to $E-T$ (recursively) or
   (b) it can be reduced to $T$

2. How can the input be reduced to the form $T$?

   (a) If the entire input consists of only $a$ then it could be reduced to $T$ or
   (b) If the entire input could be reduced to the form $(E)$ then it could be reduced to $T$.

3. How can the input be reduced to the form $E-T$?

   (a) If the entire input could be split into 3 parts $\alpha$, $\beta$ and $\gamma$ such that
       i. $\alpha$ is a prefix that can be reduced to $E$, and
       ii. $\beta = -$ , and
       iii. $\gamma$ is a suffix that can be reduced to $T$

   then it could be reduced to $E-T$.
Closures of Items

Theoretically each item is a state of a NFA. The above reasoning leads to forming closures of items to obtain DFA states, in a manner similar to the subset construction. Essentially all NFA states with similar initial behaviours are grouped together to form a single DFA state.

NFA to DFA construction

Algorithm 4 Closures of Items

Require: Set $I$ of LR(0) items of a CFG with rule set $P$
Ensure: Closure of $I$ for a subset $I \subseteq I$ of items

1. repeat
2. for all $A \rightarrow \alpha \uparrow X \beta \in I$ do
3. for all $X \rightarrow \gamma \in P$ do
4. $I := I \cup \{X \rightarrow \uparrow \gamma\}$
5. end for
6. end for
7. until no more changes occur in $I$
State Changes on Nonterminals

As in the case of the NFA to DFA construction with each state transition we also need to compute closures on the target states.

Algorithm 5 Goto for a set of $I$ of items

Require: $I \subseteq \mathcal{I}$ and $X \in N$

Ensure: States of the DFA

1. $J := \emptyset$
2. for all $A \rightarrow \alpha X \beta \in I$ do
3. \hspace{1em} $J := J \cup \{A \rightarrow \alpha X \beta\}$
4. end for
5. return Closure$(J)$
State S1

\[ S1 = \text{Closure}(\{ S \rightarrow \triangledown E$ \}) \]
\[ = \{ S \rightarrow \triangledown E$, \ E \rightarrow \triangledown E−T, \ E \rightarrow \triangledown T, \ T \rightarrow \triangledown a, \ T \rightarrow \triangledown (E) \} \]

\[ S1 \xrightarrow{E} \text{Closure}(\{ T \rightarrow (\triangledown E) \}) = S2 \]

\[ S1 \xrightarrow{T} \text{Closure}(\{ E \rightarrow T \triangledown \}) = S7 \]

\[ S1 \xrightarrow{a} \text{Closure}(\{ T \rightarrow a \triangledown \}) = S8 \]
State S2

\[ S_2 = \text{Closure}(\{ T \to (\triangledown E) \}) \]
\[ = \{ T \to \triangledown (E), E \to \triangledown E - T, E \to \triangledown T, \\
T \to \triangledown a, T \to \triangledown (E) \} \]

\[ S_2 \xrightarrow{E} \text{Closure}(\{ T \to (E \triangledown), E \to E \triangledown - T \}) = S_9 \]

\[ S_2 \xrightarrow{T} \text{Closure}(\{ E \to T \triangledown \}) = S_7 \]

\[ S_2 \xrightarrow{a} \text{Closure}(\{ T \to a \triangledown \}) = S_8 \]
Other States

\[ S3 = \text{Closure}(\{ S \to E \uparrow \$, \ E \to E \uparrow -T \}) = \{ S \to E \uparrow \$, \ E \to E \uparrow -T \} \]

However,

\[ S3 \longrightarrow \text{Closure}(\{ E \to E - \uparrow T \}) \]

and

\[ \text{Closure}(\{ E \to E - \uparrow T \}) = \{ E \to E - \uparrow T, \ T \to (\uparrow E), \ T \to \uparrow a \} = S4 \]

The closures of the other reachable sets of items are themselves.

- \[ S5 = \{ E \to E - T \uparrow \} \]
- \[ S6 = \{ S \to E \$ \uparrow \} \]
- \[ S7 = \{ E \to T \uparrow \} \]
- \[ S8 = \{ T \to a \uparrow \} \]
- \[ S9 = \{ T \to (E \uparrow), \ E \to E \uparrow -T \} \]
- \[ S10 = \{ T \to (E) \uparrow \} \]
Example: DFA
Example: Parsing Table

<table>
<thead>
<tr>
<th>States</th>
<th>Input</th>
<th>Nonterminals</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\text{a}$</td>
<td>$\text{S}$ $\text{E}$ $\text{T}$</td>
</tr>
<tr>
<td>S1</td>
<td>S8 S2</td>
<td></td>
</tr>
<tr>
<td>S2</td>
<td>S8 S2</td>
<td>G3 G7</td>
</tr>
<tr>
<td>S3</td>
<td>ACC S4</td>
<td></td>
</tr>
<tr>
<td>S4</td>
<td>S8 S2</td>
<td></td>
</tr>
<tr>
<td>S5</td>
<td>R2 R2 R2 R2 R2</td>
<td>G5</td>
</tr>
<tr>
<td>S6</td>
<td>R1 R1 R1 R1 R1</td>
<td></td>
</tr>
<tr>
<td>S7</td>
<td>R3 R3 R3 R3 R3</td>
<td></td>
</tr>
<tr>
<td>S8</td>
<td>R4 R4 R4 R4 R4</td>
<td></td>
</tr>
<tr>
<td>S9</td>
<td>S10 S4</td>
<td></td>
</tr>
<tr>
<td>S10</td>
<td>R5 R5 R5 R5 R5</td>
<td></td>
</tr>
</tbody>
</table>

Note: All empty entries denote errors
Example 6.11 Consider the following simple input viz. \textit{a$}. Here are the parsing steps.
Example 6.11 Consider the following simple input viz. a$. Here are the parsing steps.

DFA Parsing Table

| S1 | a$ | Shift S8 |
Example 6.11 Consider the following simple input viz. $a$. Here are the parsing steps.

DFA Parsing Table

<table>
<thead>
<tr>
<th>State</th>
<th>Symbol</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>a$</td>
<td>Shift S8</td>
</tr>
<tr>
<td>S1</td>
<td>a</td>
<td>S8</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>Reduce Rule 4</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>State</th>
<th>Symbol</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>a</td>
<td>S8</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>Reduce Rule 4</td>
</tr>
</tbody>
</table>
Example 6.11 Consider the following simple input viz. $a$. Here are the parsing steps.

DFA Parsing Table

<table>
<thead>
<tr>
<th>S1</th>
<th>a$</th>
<th>Shift S8</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>a</td>
<td>S8</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>Reduce Rule 4</td>
</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>$</td>
</tr>
<tr>
<td></td>
<td>$</td>
<td>Goto S7</td>
</tr>
</tbody>
</table>
Example 6.11  Consider the following simple input viz. $a\$. Here are the parsing steps.

<table>
<thead>
<tr>
<th>DFA Parsing Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1 a $</td>
</tr>
<tr>
<td>S1 a S8 $</td>
</tr>
<tr>
<td>S1 T $</td>
</tr>
<tr>
<td>S1 T S7 $</td>
</tr>
</tbody>
</table>
Example 6.11 *Consider the following simple input viz. a$. Here are the parsing steps.*

<table>
<thead>
<tr>
<th>DFA Parsing Table</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
</tr>
<tr>
<td>S1</td>
</tr>
<tr>
<td>S1</td>
</tr>
<tr>
<td>S1</td>
</tr>
<tr>
<td>S1</td>
</tr>
</tbody>
</table>
Example 6.11 Consider the following simple input viz. $a$. Here are the parsing steps.

DFA Parsing Table

<table>
<thead>
<tr>
<th>S1</th>
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<td>S1</td>
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<td>Reduce Rule 4</td>
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Example 6.12 Here is a slightly more complex input $a - (a - a)$.
**Example 6.12** Here is a slightly more complex input $a - (a - a)$. 

<table>
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<tr>
<th>State</th>
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<tbody>
<tr>
<td>S1</td>
<td>$a - (a - a)$</td>
<td>Shift S8</td>
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</table>
Example 6.12 Here is a slightly more complex input \( a - (a - a) \).
**Example 6.12** Here is a slightly more complex input $a - (a - a)$. 

<table>
<thead>
<tr>
<th>State</th>
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</thead>
<tbody>
<tr>
<td>S1</td>
<td>$a - (a - a)$</td>
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</tr>
<tr>
<td>S1</td>
<td>$a$ S8</td>
<td>$(a - a)$</td>
</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>$(a - a)$</td>
</tr>
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Example 6.12 Here is a slightly more complex input $a - (a - a)$. 

<table>
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<tr>
<th>State</th>
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<th>Action</th>
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<td>Shift S8</td>
</tr>
<tr>
<td>S1</td>
<td>S8</td>
<td>Reduce Rule 4</td>
</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>Go to S7</td>
</tr>
<tr>
<td>S1</td>
<td>T S7</td>
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### Example 6.12

Here is a slightly more complex input $a - (a - a)$.  

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<tr>
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<td>$-$</td>
<td>Shift S8</td>
</tr>
<tr>
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<td>S8</td>
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</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>Go to S7</td>
</tr>
<tr>
<td>S1</td>
<td>T S7</td>
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</tr>
<tr>
<td>S1</td>
<td>E</td>
<td>Go to S3</td>
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Example 6.12 Here is a slightly more complex input $a - (a - a)$. 

DFA Parsing Table

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<td>S8</td>
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</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>Go to S7</td>
</tr>
<tr>
<td>S1</td>
<td>T</td>
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<td>S1</td>
<td>E</td>
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</tr>
<tr>
<td>S1</td>
<td>E</td>
<td>Shift S4</td>
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Example 6.12 Here is a slightly more complex input $a - (a - a)$.

DFA Parsing Table

<table>
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<th>Action</th>
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<tbody>
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<td>$a - (a - a)$</td>
<td>Shift S8</td>
</tr>
<tr>
<td>S1</td>
<td>$a$</td>
<td>$S8$</td>
</tr>
<tr>
<td>S1</td>
<td>$T$</td>
<td>$S7$</td>
</tr>
<tr>
<td>S1</td>
<td>$T$</td>
<td>$S7$</td>
</tr>
<tr>
<td>S1</td>
<td>$E$</td>
<td>$S3$</td>
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<tr>
<td>S1</td>
<td>$E$</td>
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<tr>
<td>S1</td>
<td>$E$</td>
<td>$S3$</td>
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Example 6.12 Here is a slightly more complex input $a - (a - a)$. 

**DFA Parsing Table**

<table>
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<tr>
<th>State</th>
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<tbody>
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<td>$a - (a - a)$</td>
<td>Shift S8</td>
</tr>
<tr>
<td>S1 a</td>
<td>S8</td>
<td>Reduce Rule 4</td>
</tr>
<tr>
<td>S1 T</td>
<td>$-(a - a)$</td>
<td>Go to S7</td>
</tr>
<tr>
<td>S1 T S7</td>
<td>$-(a - a)$</td>
<td>Reduce Rule 3</td>
</tr>
<tr>
<td>S1 E</td>
<td>$-(a - a)$</td>
<td>Go to S3</td>
</tr>
<tr>
<td>S1 E S3</td>
<td>$-(a - a)$</td>
<td>Shift S4</td>
</tr>
<tr>
<td>S1 E S3 S4</td>
<td>$-(a - a)$</td>
<td>Shift S2</td>
</tr>
<tr>
<td>S1 E S3 S4 S2</td>
<td>$a - a)$</td>
<td>Shift S8</td>
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### Example 6.12

*Here is a slightly more complex input* \( a - (a - a) \).

<table>
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</thead>
<tbody>
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<td>S1</td>
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</tr>
<tr>
<td>S1 a S8</td>
<td>( -(a - a) )</td>
</tr>
<tr>
<td>S1 T</td>
<td>( -(a - a) )</td>
</tr>
<tr>
<td>S1 T S7</td>
<td>( -(a - a) )</td>
</tr>
<tr>
<td>S1 E</td>
<td>( -(a - a) )</td>
</tr>
<tr>
<td>S1 E S3</td>
<td>( -(a - a) )</td>
</tr>
<tr>
<td>S1 E S3 – S4</td>
<td>( a - a )</td>
</tr>
<tr>
<td>S1 E S3 – S4 ( S2</td>
<td>a</td>
</tr>
<tr>
<td>S1 E S3 – S4 ( S2 a S8</td>
<td>( -a )</td>
</tr>
</tbody>
</table>
Example 6.12 Here is a slightly more complex input $a - (a - a)$.

<table>
<thead>
<tr>
<th>State</th>
<th>Input</th>
<th>Action</th>
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</thead>
<tbody>
<tr>
<td>S1</td>
<td>$a - (a - a)$</td>
<td>Shift S8</td>
</tr>
<tr>
<td>S1</td>
<td>$a$</td>
<td>$(a - a)$</td>
</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>$(a - a)$</td>
</tr>
<tr>
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<td>T S7</td>
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<tr>
<td>S1</td>
<td>E</td>
<td>$(a - a)$</td>
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<td>E S3</td>
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Example 6.12 Here is a slightly more complex input $a - (a - a)$.

DFA Parsing Table

<table>
<thead>
<tr>
<th>State</th>
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<th>Action</th>
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<td>$a$ S8</td>
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</tr>
<tr>
<td>S1</td>
<td>T</td>
<td>Go to S7</td>
</tr>
<tr>
<td>S1</td>
<td>T S7</td>
<td>Reduce Rule 3</td>
</tr>
<tr>
<td>S1</td>
<td>E</td>
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</tr>
<tr>
<td>S1</td>
<td>E S3</td>
<td>Shift S4</td>
</tr>
<tr>
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<td>(a – a) $</td>
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$\text{Go to S7}$

Reduce Rule 3
Example 6.12 Here is a slightly more complex input $a - (a - a)$. 

DFA Parsing Table

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DFA Parsing Table
DFA Parsing Table

| S1 | E | S3 | − | S4 | ( | S2 | E | S9 | − | S4 | | −a)($) | Shift S4 |
| S1 | E | S3 | − | S4 | ( | S2 | E | S9 | − | S4 | | a($) | Shift S8 |
DFA Parsing Table

S1 E S3 → S4 ( S2 E S9 ) −a)§ Shift S4

S1 E S3 → S4 ( S2 E S9 − S4 ) a)§ Shift S8

S1 E S3 → S4 ( S2 E S9 − S4 a S8 )§ Reduce Rule 4
### DFA Parsing Table

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### DFA Parsing Table

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<tbody>
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<td>State 4</td>
<td>(</td>
<td>State 2</td>
<td>E</td>
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<tr>
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<td>State 4</td>
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<td>−</td>
<td>State 4</td>
<td>(</td>
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<td>−</td>
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<td>−</td>
<td>State 4</td>
<td>(</td>
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<td>State 9</td>
<td>−</td>
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</table>
DFA Parsing Table

S1 E S3 − S4 ( S2 E S9 − S4 −a)$ Shift S4

S1 E S3 − S4 ( S2 E S9 − S4 a S8 )$ Reduce Rule 4

S1 E S3 − S4 ( S2 E S9 − S4 a S8 )$ Go to S5

S1 E S3 − S4 ( S2 E S9 − S4 T S5 )$ Reduce Rule 2

S1 E S3 − S4 ( S2 E S9 − S4 )$ Go to S9

S1 E S3 − S4 ( S2 E S9 − S4 S10 )$ Shift S10
### DFA Parsing Table

<table>
<thead>
<tr>
<th>State</th>
<th>Symbol</th>
<th>Next State</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td>S1</td>
<td>E</td>
<td>S3 − S4</td>
</tr>
<tr>
<td></td>
<td>−a$</td>
<td>Shift S4</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td>a$</td>
<td>Shift S8</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td>S8</td>
<td>Reduce Rule 4</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td>T</td>
<td>Go to S5</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td>T S5</td>
<td>Reduce Rule 2</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Go to S9</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Shift S10</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>S4</td>
</tr>
<tr>
<td></td>
<td>S10</td>
<td>Reduce Rule 5</td>
</tr>
</tbody>
</table>

**Rule 4:**
- S1 E S3 − S4 (S2 E S9 − S4 T S5) $ Reduce Rule 2

**Rule 5:**
- S1 E S3 − S4 (S2 E S9) S10 $ Reduce Rule 5
DFA Parsing Table

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & ) & a & $ & \text{Shift S4} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & - & S4 & ) & a & $ & \text{Shift S8} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & - & S4 & a & S8 & ) & $ & \text{Reduce Rule 4} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & - & S4 & T & ) & $ & \text{Go to S5} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & - & S4 & T & S5 & ) & $ & \text{Reduce Rule 2} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & ) & $ & \text{Go to S9} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & ) & $ & \text{Shift S10} \\
\hline
S1 & E & S3 & - & S4 & ( & S2 & E & S9 & ) & S10 & $ & \text{Reduce Rule 5} \\
\hline
S1 & E & S3 & - & S4 & T & $ & \text{Go to S5} \\
\hline
\end{array}
\]
### DFA Parsing Table

<table>
<thead>
<tr>
<th>State</th>
<th>Symbol</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>E</td>
<td>Shift S4</td>
</tr>
<tr>
<td>S1</td>
<td>S3</td>
<td>Shift S8</td>
</tr>
<tr>
<td>S1</td>
<td>S4</td>
<td>Reduce Rule 4</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4</td>
<td>Go to S5</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4</td>
<td>Reduce Rule 2</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4</td>
<td>Go to S9</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4</td>
<td>Shift S10</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4</td>
<td>Reduce Rule 5</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4</td>
<td>Go to S5</td>
</tr>
<tr>
<td>S1</td>
<td>S3, S4, S5</td>
<td>Reduce Rule 2</td>
</tr>
</tbody>
</table>
### DFA Parsing Table

<table>
<thead>
<tr>
<th>State</th>
<th>Symbol</th>
<th>Action</th>
</tr>
</thead>
<tbody>
<tr>
<td>S1</td>
<td>E</td>
<td>S3</td>
</tr>
<tr>
<td>S3</td>
<td>S4</td>
<td>S4</td>
</tr>
<tr>
<td>S4</td>
<td>(a)</td>
<td><strong>Reduce Rule 4</strong></td>
</tr>
<tr>
<td>S4</td>
<td>S8</td>
<td><strong>Reduce Rule 4</strong></td>
</tr>
<tr>
<td>S8</td>
<td>T</td>
<td>Go to S5</td>
</tr>
<tr>
<td>S8</td>
<td>S5</td>
<td><strong>Reduce Rule 2</strong></td>
</tr>
<tr>
<td>S5</td>
<td>T</td>
<td>Go to S5</td>
</tr>
<tr>
<td>S5</td>
<td>S10</td>
<td><strong>Reduce Rule 5</strong></td>
</tr>
<tr>
<td>S10</td>
<td>T</td>
<td>Go to S5</td>
</tr>
<tr>
<td>S10</td>
<td>S5</td>
<td><strong>Reduce Rule 2</strong></td>
</tr>
<tr>
<td>S5</td>
<td>E</td>
<td>Go to S3</td>
</tr>
</tbody>
</table>
DFA Parsing Table

S1 E S3 − S4 ( S2 E S9 − S4 −a)Δ $ Shift S4

S1 E S3 − S4 ( S2 E S9 − S4 a S8 )Δ $ Reduce Rule 4

S1 E S3 − S4 ( S2 E S9 − S4 a S8 )Δ $ Go to S5

S1 E S3 − S4 ( S2 E S9 − S4 T S5 )Δ $ Reduce Rule 2

S1 E S3 − S4 ( S2 E S9 )Δ $ Go to S9

S1 E S3 − S4 ( S2 E S9 )Δ $ Shift S10

S1 E S3 − S4 ( S2 E S9 ) S10 Δ $ Reduce Rule 5

S1 E S3 − S4 T Δ $ Go to S5

S1 E S3 − S4 T S5 Δ $ Reduce Rule 2

S1 E Δ $ Go to S3

S1 E S3 Δ $ Accept
Exercise 6.2

1. Design a LR(0) parser for the grammar of **palindromes**. Identify whether there are any conflicts in the parsing table.

2. Design a LR(0) parser for the grammar of **Matching brackets** and identify any conflicts.

3. Design a context-free grammar for a language on the terminal symbols \(a\) and \(b\) such that every string has more \(a\)s than \(b\)s. Design a LR(0) parser for this grammar and find all the conflicts, if any.

4. Since every regular expression may also be represented by a context-free grammar design an LR(0) parser for comments in C.
CFG = RLG + Bracket Matching

We use the idea that a context-free grammar is essentially a regular grammar with parentheses matching to arbitrary depths. Hence a DFA with some reductions introduced may work. We modify the grammar to have a special terminal symbol called the end-marker (denoted by $). Now consider the following simple grammar with a single right-associative binary operator $\hat{\text{\textdagger}}$ and bracket-matching. We create a DFA of “items” which also have a special marker called the “cursor” (▲).
LR(0) with Right-Association

Consider the following grammar

1. \( S \rightarrow E$ 
2. \( E \rightarrow P ^ E \)
3. \( E \rightarrow P \)
4. \( P \rightarrow a \)
5. \( P \rightarrow (E) \)

The following items make up the initial state \( S_1 \) of the DFA

\[ I_{1.1} \quad S \rightarrow \Delta E$ \]
\[ I_{2.1} \quad E \rightarrow \Delta P ^ E \]
\[ I_{3.1} \quad E \rightarrow \Delta P \]
\[ I_{4.1} \quad P \rightarrow \Delta a \]
\[ I_{5.1} \quad P \rightarrow \Delta(E) \]
Shift-Reduce Conflicts in LR(0)

There is a transition on the nonterminal $P$ to the state $S2$ which is made up of the following items.

$$I2.2 \ E \rightarrow \ P \uparrow \hat{E}$$
$$I3.2 \ E \rightarrow \ P \uparrow$$

Then clearly the LR(0) parser suffers a *shift-reduce* conflict because

- item I2.2 indicates a *shift* action,
- item I3.2 produces a *reduce* action

This in contrast to the parsing table *produced earlier* where reduce actions took place regardless of the input symbol. Clearly now that principle will have to be modified.

The parsing table in this case would have a *shift* action if the input in state $S2$ is a $\hat{\imath}$ and a *reduce* action for all other input symbols.
FOLLOW Sets

We construct for each non-terminal symbol a set of terminal symbols that can *follow* this non-terminal in any rightmost derivation. In the previous grammar we have

\[
\text{FOLLOW}(E) = \{\$, \)}
\]
\[
\text{FOLLOW}(P) = \{^\}\}
\]

Depending upon the input symbol and whether it appears in the FOLLOW set of the non-terminal under question we resolve the shift-reduce conflict.

This modification to LR(0) is called **Simple LR (SLR)** parsing method. However SLR is not powerful enough for many useful grammar constructions that are encountered in many programming languages.
Computing FIRST Sets

In order to compute FOLLOW sets we require FIRST sets of sentential forms to be constructed too.

1. FIRST \((a)\) = \(\{a\}\) for every terminal symbol \(a\).
2. \(\varepsilon \in \text{FIRST}(X)\) if \(X \rightarrow \varepsilon \in P\).
3. If \(X \rightarrow Y_1Y_2\cdots Y_k \in P\) then \(\text{FIRST}(Y_1) \subseteq \text{FIRST}(X)\)
4. If \(X \rightarrow Y_1Y_2\cdots Y_k \in P\) then for each \(i : i < k\) such that \(Y_1Y_2\cdots Y_i \Rightarrow \varepsilon\), \(\text{FIRST}(Y_{i+1}) \subseteq \text{FIRST}(X)\).
Computing FOLLOW Sets

Once FIRST has been computed, computing FOLLOW for each non-terminal symbol is quite easy.

1. $\$ \in \text{FOLLOW}(S)$ where $S$ is the start symbol of the augmented grammar.

2. For each production rule of the form $A \rightarrow \alpha B \beta$, $\text{FIRST}(\beta) - \{\varepsilon\} \subseteq \text{FOLLOW}(B)$.

3. For each production rule of the form $A \rightarrow \alpha B \beta$, if $\varepsilon \in \text{FIRST}(\beta)$ then $\text{FOLLOW}(A) \subseteq \text{FOLLOW}(B)$.

4. For each production of the form $A \rightarrow \alpha B$, $\text{FOLLOW}(A) \subseteq \text{FOLLOW}(B)$. 
if-then-else vs. if-then

Most programming languages have two separate constructs if-then and if-then-else. We abbreviate the keywords and use the following symbols

<table>
<thead>
<tr>
<th>Tokens</th>
<th>Symbols</th>
</tr>
</thead>
<tbody>
<tr>
<td>if</td>
<td>i</td>
</tr>
<tr>
<td>then</td>
<td>t</td>
</tr>
<tr>
<td>else</td>
<td>e</td>
</tr>
<tr>
<td>booleans</td>
<td>b</td>
</tr>
<tr>
<td>other expressions</td>
<td>a</td>
</tr>
</tbody>
</table>

and construct the following two augmented grammars $G_1$ and $G_2$.

1. $S \rightarrow I \;\$ 
2. $I \rightarrow U$ 
3. $I \rightarrow M$ 
4. $U \rightarrow ibt \; I$ 
5. $U \rightarrow ibt \; M \; e \; U$ 
6. $M \rightarrow ibt \; M \; e \; M$ 
7. $M \rightarrow a$ 

1. $S \rightarrow I \;\$ 
2. $I \rightarrow ibt \; I \; E$ 
3. $I \rightarrow a$ 
4. $E \rightarrow e \; I$ 
5. $E \rightarrow \varepsilon$ 
6. $E \rightarrow \varepsilon$
Problems in LR parsing

1. Prove that grammar $G_2$ is ambiguous.

2. Construct the LR(0) parsing tables for both $G_1$ and $G_2$ and find all shift-reduce conflicts in the parsing table.

3. Construct the FOLLOW sets in each case and try to resolve the conflicts.

4. Show that the following augmented grammar cannot be parsed (i.e. there are conflicts that cannot be resolved by FOLLOW sets) either by LR(0) or SLR parsers. (*Hint* First construct the LR(0) DFA).

1. $S \rightarrow E\$
2. $E \rightarrow L = R$
3. $E \rightarrow R$
4. $L \rightarrow * R$
5. $L \rightarrow a$
6. $R \rightarrow L$
nullable

A nonterminal symbol $X$ is **nullable** if it can derive the empty string, i.e. $X \Rightarrow^* \varepsilon$.

---

**Algorithm 6 Nullable**

**Require:** CFG $G = (N, T, P, S)$

**Ensure:** $NULLABLE(N \cup T)$

1. for all $\alpha \in N \cup T$ do
2.   if $\alpha \rightarrow \varepsilon \in P$ then
3.     $NULLABLE(\alpha) := true$
4.   else
5.     $NULLABLE(\alpha) := false$
6.   end if
7. end for
8. repeat
9.   for all $X \rightarrow \alpha_1 \ldots \alpha_k \in P$ do
10.      if $\forall i : 1 \leq i \leq k : NULLABLE(\alpha_i)$ then
11.         $NULLABLE(X) := true$
12.      end if
13. end for
14. until $NULLABLE(N \cup T)$ is unchanged
First

\( \textsc{First}(\alpha) \) is the set of terminal symbols that can be the first symbol of any string that \( \alpha \) can derive, i.e. \( a \in \textsc{First}(\alpha) \) if and only if there exists a derivation \( \alpha \Rightarrow^* ax \) for any string of terminals \( x \).

Algorithm 7 First

Require: CFG \( G = \langle N, T, P, S \rangle \)
Ensure: \( \textsc{First}(N \cup T) \)

1. for all \( a \in T \) do
2. \( \textsc{First}(a) := \{a\} \)
3. end for
4. for all \( X \in N \) do
5. \( \textsc{First}(X) := \emptyset \)
6. end for
7. repeat
8. for all \( X \rightarrow \alpha_1 \ldots \alpha_k \in P \) do
9. \( \text{for } i := 1 \ldots k \) do
10. \( \text{if } \forall i' : 1 \leq i' < i : \text{NULLABLE}(\alpha_{i'}) \text{ then} \)
11. \( \textsc{First}(X) := \textsc{First}(X) \cup \textsc{First}(\alpha_i) \)
12. \( \text{end if} \)
13. \( \text{end for} \)
14. \( \text{end for} \)
15. until \( \textsc{First}(N \cup T) \) sets are all unchanged
First And Follow

Notice that if \( X \rightarrow \alpha Z \beta \) is a production then one cannot ignore the \( FIRST(Z) \) in computing \( FIRST(X) \) especially if \( \alpha \Rightarrow^* \varepsilon \). Further if \( Z \) is also nullable then \( FIRST(\beta) \subseteq FIRST(X) \).

\( FOLLOW(X) \) for any nonterminal symbol \( X \) is the set of terminal symbols \( a \) such that there exists a rightmost derivation of the form

\[
S \Rightarrow^* \cdots X a \cdots \Rightarrow^*
\]

i.e. \( FOLLOW(X) \) is the set of all terminal symbols that can occur to the right of \( X \) in a rightmost derivation.

Notice that if there exists a a rightmost derivation of the form

\[
S \Rightarrow^* \cdots X \alpha_1 \cdots \alpha_k a \cdots \Rightarrow^*
\]

such that \( \alpha_1, \ldots, \alpha_k \) are all nullable then again we have

\[
S \Rightarrow^* \cdots X \alpha_1 \cdots \alpha_k a \cdots \Rightarrow^* \cdots X a \cdots \Rightarrow^*
\]
Computing Follow

Algorithm 8 Follow

Require: CFG $G = \langle N, T, P, S \rangle$
Ensure: $\text{FOLLOW}(N)$

1. for all $\alpha \in N \cup T$ do
2. \hspace{1em} $\text{FOLLOW}(\alpha) := \emptyset$
3. end for
4. repeat
5. \hspace{1em} for all $X \to \alpha_1 \ldots \alpha_k \in P$ do
6. \hspace{2em} for $i := 1 \ldots k$ do
7. \hspace{3em} if $\forall i' : i + 1 \leq i' \leq k : \text{NULLABLE}(\alpha_{i'})$ then
8. \hspace{4em} $\text{FOLLOW}(\alpha_i) := \text{FOLLOW}(\alpha_i) \cup \text{FOLLOW}(X)$
9. \hspace{3em} end if
10. \hspace{2em} for $j := i + 1 \ldots k$ do
11. \hspace{3em} if $\forall i' : i + 1 \leq i' \leq j - 1 : \text{NULLABLE}(\alpha_{i'})$ then
12. \hspace{4em} $\text{FOLLOW}(\alpha_i) := \text{FOLLOW}(\alpha_i) \cup \text{FIRST}(\alpha_j)$
13. \hspace{3em} end if
14. \hspace{2em} end for
15. \hspace{1em} end for
16. end for
17. until $\text{FOLLOW}(N \cup T)$ sets are all unchanged
Recursive Descent Parsing

• Suitable for grammars that are LL(1)
• A set of (mutually) recursive procedures
• Has a single procedure/function for each non-terminal symbol
• Allows for syntax errors to be pinpointed more accurately than most other parsing methods
Caveats with RDP: Left Recursion

Any direct or indirect left-recursion in the grammar can lead to infinite recursive calls during which no input token is consumed and there is no return from the recursion. In particular,

• Production rules cannot be left-recursive i.e. they should not be of the form $A \rightarrow A\alpha$. This would result in an infinite recursion with no input token consumed.

• A production cannot even be indirectly left recursive. For instance the following is indirect left-recursion of cycle length 2.

Example 6.13

$$A \rightarrow B\beta$$
$$B \rightarrow A\alpha$$

where $\alpha, \beta \in (N \cup T)^*$. 

• In general it should be impossible to have derivation sequences of the form $A \Rightarrow A_1\alpha_1 \cdots \Rightarrow A_{n-1}\alpha_{n-1} \Rightarrow A\alpha_n$ for nonterminal symbols $A, A_1, \ldots, A_{n-1}$ for any $n > 0$. 
Caveats with RDP: Left Factoring

For RDP to succeed without backtracking, for each input token and each non-terminal symbol there should be only one rule applicable;

**Example 6.14** A set of productions of the form

\[ A \rightarrow aB\beta | aC\gamma \]

where \( B \) and \( C \) stand for different phrases would lead to non-determinism. The normal practice then would be to left-factor the two productions by introducing a new non-terminal symbol \( A' \) and rewrite the rule as

\[ A \rightarrow aA' \]

\[ A' \rightarrow B\beta | C\gamma \]

provided \( B \) and \( C \) generate terminal strings with different first symbols (otherwise more left-factoring needs to be performed).
Left Recursion

The grammar used in shift-reduce parsing is clearly left-recursive in both the nonterminals $E$ and $T$ and hence is not amenable to recursive-descent parsing.

The grammar may then have to be modified as follows:

$$
E \rightarrow TE' \\
E' \rightarrow -TE' \mid \varepsilon \\
T \rightarrow DT' \\
T' \rightarrow /DT' \mid \varepsilon \\
D \rightarrow a \mid b \mid (E)
$$

Now this grammar is no longer left-recursive and may then be parsed by a recursive-descent parser.

Specification of Syntax: EBNF
6.8.1. The Extended Backus-Naur Form (EBNF)

The EBNF specification of a programming language is a collection of rules that defines the (context-free) grammar of the language. It specifies the formation rules for the correct grammatical construction of the phrases of the language.

**Start symbol.** The rules are written usually in a “top-down fashion” and the very first rule gives the productions of the start symbol of the grammar.

**Non-terminals.** Uses English words or phrases to denote non-terminal symbols. These words or phrases are suggestive of the nature or meaning of the constructs.

**Metasymbols.**

- Sequences of constructs enclosed in “{” and “}” denote zero or more occurrences of the construct (c.f. Kleene closure on regular expressions).
- Sequences of constructs enclosed in “[” and “]” denote that the enclosed constructs are optional i.e. there can be only zero or one occurrence of the sequence.
- Constructs are enclosed in “(” and “)” to group them together.
- “|” separates alternatives.
- “::=” defines the productions of each non-terminal symbol.
- “.” terminates the possibly many rewrite rules for a non-terminal.

**Terminals.** Terminal symbol strings are usually enclosed in double-quotes when written in monochrome (we shall additionally colour-code them).
Balanced Parentheses: CFG

Example 6.15 A context-free grammar for balanced parentheses (including the empty string) over the terminal alphabet \{ (, ), [ , ], { , } \} could be given as \( BP_3 = \langle \{ S \}, \{ (, ), [ , ], { , } \}, P, \{ S \} \rangle \), where \( P \) consists of the productions

\[
\begin{align*}
S & \rightarrow \epsilon, \\
S & \rightarrow (S)S, \\
S & \rightarrow [S]S, \\
S & \rightarrow \{S\}S
\end{align*}
\]
Balanced Parentheses: EBNF

Example 6.16 $BP_3$ may be expressed in EBNF as follows:

\[
\begin{align*}
\text{BracketSeq} &::= \{ \text{Bracket} \} . \\
\text{Bracket} &::= \text{LeftParen} \text{BracketSeq} \text{RightParen} | \\
&| \text{LeftSqbracket} \text{BracketSeq} \text{RightSqbracket} | \\
&| \text{LeftBrace} \text{BracketSeq} \text{RightBrace} .
\end{align*}
\]

\[
\begin{align*}
\text{LeftParen} &::= “(” . \\
\text{RightParen} &::= “)” . \\
\text{LeftSqbracket} &::= “[” . \\
\text{RightSqbracket} &::= “]” . \\
\text{LeftBrace} &::= “{” . \\
\text{RightBrace} &::= “}” .
\end{align*}
\]
EBNF has its own grammar which is again context-free. Hence EBNF (6.8.1) may be used to define EBNF in its own syntax as follows:

Syntax ::= {Production}.
Production ::= NonTerminal “::=” PossibleRewrites “.”.
PossibleRewrites ::= Rewrite {“|” Rewrite}.
Rewrite ::= Symbol {Symbol}.
Symbol ::= NonTerminal | Terminal | GroupRewrites.
GroupRewrites ::= “{” PossibleRewrites “}” | “[” PossibleRewrites “]” | “(” PossibleRewrites “)”.
NonTerminal ::= Letter {Letter | Digit}.
Terminal ::= Character {Character}.
EBNF: Character Set

The character set used in EBNF is described below.

Character ::= Letter | Digit | SpecialChar
Letter ::= UpperCase | LowerCase
UpperCase ::= “A” | “B” | “C” | “D” | “E” | “F” | “G” | “H” |
           | “I” | “J” | “K” | “L” | “M” | “N” | “O” | “P” | “Q” |
LowerCase ::= “a” | “b” | “c” | “d” | “e” | “f” | “g” | “h” |
            | “i” | “j” | “k” | “l” | “m” | “n” | “o” | “p” | “q” |
            | “r” | “s” | “t” | “u” | “v” | “w” | “x” | “y” | “z”
Digit ::= “0” | “1” | “2” | “3” | “4” | “5” | “6” | “7” | “8” | “9”
SpecialChar ::= “!” | “” | “#” | “$” | “%” | “&” | “’” | “(” | “)” | “*” |
               | “+” | “,” | “-” | “.” | “/” | “:” | “;” | “<” | “=” | “>” | “?”
               | “@” | “[” | “\” | “]” | “^” | “_” | “_” | “_” | “{” | “|” | “}” | “”
All words written in **bold** font are reserved words and cannot be used as identifiers in any program.

```
Program ::= "program" Identifier "::" Block .
Block ::= DeclarationSeq CommandSeq .
DeclarationSeq ::= {Declaration} .
Declaration ::= "var" VariableList "::" Type ";" .
Type ::= "int" | "bool" .
VariableList ::= Variable {"," Variable} .
CommandSeq ::= "{" {Command ";" } "}" .
Command ::= Variable "::=" Expression | 
         "read" Variable | 
         "write" IntExpression | 
         "if" BoolExpression 
         "then" CommandSeq 
         "else" CommandSeq 
         "endif" | 
         "while" BoolExpression "do" 
         CommandSeq 
         "endwh" .
```
Expression ::= IntExpression | BoolExpression.

IntExpression ::= IntExpression AddOp IntTerm | IntTerm.

IntTerm ::= IntTerm MultOp IntFactor | IntFactor.

IntFactor ::= Numeral | Variable | 
              ("IntExpression") | ~ IntFactor.

BoolExpression ::= BoolExpression || BoolTerm | BoolTerm.

BoolTerm ::= BoolTerm && BoolFactor | BoolFactor.

BoolFactor ::= tt | ff | Variable | Comparison | 
              ("BoolExpression") | ! BoolFactor.

Comparison ::= IntExpression RelOp IntExpression.

Variable ::= Identifier.

RelOp ::= < | <= | = | > | >= | <>.

AddOp ::= + | -. 

MultOp ::= * | / | %.

Identifier ::= Letter {Letter | Digit}.

Numeral ::= ["+" | "]Digit {Digit}.

Note

1. ";" acts as a terminator for both Declarations and Commands.

2. "," acts as a separator in VariableList.

3. Comparison has a higher precedence than BoolTerm and BoolExpression.

4. RelOps have lower precedence than any of the integer operations specified in MultOp and AddOp.

5. The nonterminals Letter and Digit are as specified earlier in the EBNF character set.
Syntax Diagrams

• EBNF was first used to define the grammar of ALGOL-60 and the syntax was used to design the parser for the language.

• EBNF also has a diagrammatic rendering called syntax diagrams or railroad diagrams. The grammar of SML has been rendered by a set of syntax diagrams.

• Pascal has been defined using both the text-version of EBNF and through syntax diagrams.

• While the text form of EBNF helps in parsing, the diagrammatic rendering is only for the purpose of readability.

• EBNF is a specification language that almost all modern programming languages use to define the grammar of the programming language.
Syntax Specifications

• BNF of C
• BNF of Java
• EBNF of Pascal
• Pascal Syntax diagrams
• BNF of Standard ML
• BNF of Datalog
• BNF of Prolog
Syntax of Standard ML

Tobias Niblow and Larry Paulson

PROGRAMS AND MODULES

Program

TopLevelDeclaration

TopLevelDeclaration

Expression

ObjectDeclaration

SignatureDeclaration

FunctorDeclaration

ObjectDeclaration

Declaration

structure

local

ObjectDeclaration

in

ObjectDeclaration
DECLARATIONS

Declaration

val / Pattern ➔ Expression

rec and

fun FunHeading ➔ Expression

and

Type

l

Type

and

type TypeBinding

datatype DatatypeBinding

withtype TypeBinding

abstype DatatypeBinding

withtype TypeBinding

with Declaration end

exception Name of Type

- CompoundName

and

local Declaration in Declaration end

open CompoundIdent

infix

infixr

Digit

nonfix

Ident
Syntax Diagrams of SML: 5

Type Binding

```
TypeVarList ─ Ident = Type
  and
```

Datatype Binding

```
TypeVarList ─ Ident = Ident of Type
  and
```

Type Var List

```
TypeVar
  (TypeVar)
```

Atomic Pattern ─ Infix Operator ─ Atomic Pattern
EXPRESSIONS

Expression

InfixExpression

AtomicExpression

InfixOperator

InfixExpression
MATCHES AND PATTERNS

Match

Pattern $\Rightarrow$ Expression

Pattern

AtomicPattern

CompoundName AtomicPattern

Pattern InfixOperator Pattern

Pattern Type Pattern

Name Pattern
Syntax Diagrams of SML: 8

1. AtomicPattern

2. CompoundName

3. Constant

4. Pattern

5. FieldPattern

FieldPattern

- Label

- Ident

- Type

- as

- Pattern

- FieldPattern
TYPES

Type

CompoundIdent

LEXICAL MATTERS: IDENTIFIERS, CONSTANTS, COMMENTS

CompoundIdent

CompoundName

Name

InfixOperator
Syntax Diagrams of SML: 10

1. Constant
   - Numeral
   - .
   - /
   - /...
   - E
   - /
   - /
   - /
   - /
   - /
   - /
   - /
   - /
   - /
   - /
   - any printable character except \ and "
   - StringEscape

2. StringEscape
   - 2
   - 3
   - one of @ABCDEFGHIJKLMNOPQRSTUVWXYZ_-
   - Digit
   - Digit
   - Digit
   - "
   - /
   - /
   - /
   - /
   - /
   - /
   - /
   - /
   - /

3. Numeral
   - Digit
   - Digit

4. Type Var
   - AlphanumericIdent
Syntax Diagrams of SML: 11

**Label**

- **Ident**
- **Digit**

**AlphanumericIdent**

- **Letter**
- **Letter**
- **Digit**
- **-**
- **'**

**Digit**

- *one of 0123456789*

**Letter**

- *one of ABCDEFGHIJKLMNOPQRSTUVWXYZabcdefghijklmnopqrstuvwxyz*

**Comment**

- (*)
- *(any text that does not include ( * or *) as a substring)*
Exercise 6.3

1. Translate all the context-free grammars that we have so far seen into EBNF specifications.

2. Specify the language of regular expressions over a non-empty finite alphabet $A$ in EBNF.

3. Given a textual EBNF specification write an algorithm to render each non-terminal as a syntax diagram.
Attributes & Semantic Analysis
7.1. Context-sensitive analysis and Semantics

The parser for a context-free grammar transforms the token stream into a derivation tree (which we also call a concrete parse tree). What we actually require in order to perform a computation is really an abstract syntax tree.

Example 7.1 consider the two sentences $a - a/b$ and $a - (a/b)$ which are both valid sentences generated by the grammar of our favourite example.

The (possibly modified grammar) required for parsing

---

The term “parse tree” is a much abused term used to refer to anything from a derivation tree to an abstract syntax tree (AST).
• treats all tokens uniformly since the phrase structure of the grammar is all-important during the parsing process,
• introduces bracketing and punctuation marks for
  – disambiguation and
  – to facilitate easy parsing

But these symbols do not by themselves carry any semantic information.

• also has many more non-terminal symbols that are required for parsing, but which carry no semantic signifi-
cance
  – either for the end-user of the language
  – or for the later phases of the compilation process.

Both expressions in example eg:concrete-parse-trees have the same meaning (semantics). But the sentences are
syntactically different and correspondingly have different parse trees. Actually both the expressions may be repre-
sented by the following abstract syntax tree (AST) which gives the hierarchical structure of the expression.

![Abstract syntax tree](image)

Figure 2: Abstract syntax tree (AST) for the sentences in fig. 1

Notice that the AST in figure 2
• abstracts away from non-terminals which have significance only for the parsing of the expression and have no semantic significance whatsoever,

• abstracts away from bracketing and punctuation mechanisms and provides a hierarchical structure containing only the essential operators and operands.

• clearly distinguishes the operators (based on their arity) from the operands (which are leaves of the AST).
1. Every programming language can be used to program any computable function, assuming of course, it has
   - unbounded memory, and
   - unbounded time

2. Context-free grammars are not powerful enough to represent all computable functions.

   **Example 7.2** The language \( \{a^n b^n c^n | n > 0\} \) is not context-free.

3. Semantic analysis is an essential step to generating IR-code, since it requires the computation of certain *bits and pieces of information* called attributes (which include information to be entered into the symbol table or useful for error-handling)

4. Many of these attributes are *context-sensitive* in nature. They need to be computed and if necessary propagated during parsing from wherever they are available.
Semantic Analysis: 1

The parser of a programming language provides the framework within which the IR-code or even the target code is to be generated.

The parser also provides a structuring mechanism that divides the task of code generation into bits and pieces determined by the individual nonterminals and production rules.

The parser provides the framework from within which the semantic analysis (which includes the bits and pieces of information that are required for code generation) is performed.
Semantic Analysis: 2

• There are context-sensitive aspects of a program that cannot be represented/enforced by a context-free grammar definition. Examples include
  – type consistency between declaration and use.
  – correspondence between formal and actual parameters (example 7.2 is an abstraction where $a^n$ represents a function or procedure declaration with $n$ formal parameters and $b^n$ and $c^n$ represent two calls to the same procedure in which the number of actual parameters should equal $n$).
  – scope and visibility issues with respect to identifiers in a program.
7.2. Syntax-Directed Translation

Syntax-directed Translation
Attributes

An attribute can represent anything we choose e.g.

• a string

• a number (e.g. size of an array or the number of formal parameters of a function)

• a type

• a memory location

• a procedure to be executed

• an error message to be displayed

The value of an attribute at a parse-tree node is defined by the semantic rule associated with the production used at that node.
Syntax-Directed Definitions (SDD)

Syntax-Directed definitions are high-level specifications which specify the evaluation of

1. various attributes

2. various procedures such as
   • transformations
   • generating code
   • saving information
   • issuing error messages

They hide various implementation details and free the compiler writer from explicitly defining the order in which translation, transformations, and code generation take place.
Kinds of Attributes

There are two kinds of attributes that one can envisage.

**Synthesized attributes** A synthesized attribute is one whose value depends upon the values of its immediate children in the concrete parse tree.

A syntax-directed definition that uses only synthesized attributes is called an *S-attributed* definition. See example

**Inherited attributes** An inherited attribute is one whose value depends upon the values of the attributes of its parents or siblings in the parse tree.

Inherited attributes are convenient for expressing the dependence of a language construct on the *context* in which it appears.
What is Syntax-directed?

• A syntax-directed definition is a generalisation of a context-free grammar in which each grammar symbol has an associated set of attributes, partitioned into two subsets called synthesized and inherited attributes.

• The various attributes are computed by so-called semantic rules associated with each production of the grammar which allows the computation of the various attributes.

• These semantic rules are in general executed during
  bottom-up (SR) parsing at the stage when a reduction needs to be performed by the given rule and
top-down (RDP) parsing in the procedure before the next call or return from the procedure.

• A parse tree showing the various attributes at each node is called an annotated parse tree.
Forms of SDDs

In a syntax-directed definition, each grammar production rule $X \rightarrow \alpha$ has associated with it a set of semantic rules of the form $b = f(a_1, \ldots, a_k)$ where $a_1, \ldots, a_k$ are attributes belonging to $X$ and/or the grammar symbols of $\alpha$.

**Definition 7.3** Given a production $X \rightarrow \alpha$, an attribute $a$ is **synthesized**: a synthesized attribute of $X$ (denoted $X.a$) or **inherited**: an inherited attribute of one of the grammar symbols of $\alpha$ (denoted $B.a$ if $a$ is an attribute of $B$).

In each case the attribute $a$ is said to depend upon the attributes $a_1, \ldots, a_k$. 
Attribute Grammars

• An attribute grammar is a syntax-directed definition in which the functions in semantic rules can have no side-effects.

• The attribute grammar also specifies how the attributes are propagated through the grammar, by using graph dependency between the productions.

• In general different occurrences of the same non-terminal symbol in each production will be distinguished by appropriate subscripts when defining the semantic rules associated with the rule.

The following example illustrates the concept of a syntax-directed definition using synthesized attributes.
Attribute Grammars: Example

Determining the values of arithmetic expressions. Consider a simple attribute $\texttt{val}$ associated with an expression

\[
E_0 \rightarrow E_1 - T \quad \triangleright \quad E_0.\texttt{val} := E_1.\texttt{val} - T.\texttt{val}
\]

\[
E \rightarrow T \quad \triangleright \quad E.\texttt{val} := T.\texttt{val}
\]

\[
T_0 \rightarrow T_1 / F \quad \triangleright \quad T_0.\texttt{val} := T_1.\texttt{val} / F.\texttt{val}
\]

\[
T \rightarrow F \quad \triangleright \quad T.\texttt{val} := F.\texttt{val}
\]

\[
F \rightarrow (E) \quad \triangleright \quad F.\texttt{val} := E.\texttt{val}
\]

\[
F \rightarrow n \quad \triangleright \quad F.\texttt{val} := n.\texttt{val}
\]

Note: The attribute $\texttt{n.val}$ is the value of the numeral $\texttt{n}$ computed during scanning (lexical analysis).
Attributes: Basic Assumptions

• Terminal symbols are assumed to have only synthesized attributes. Their attributes are all supplied by the lexical analyser during scanning.

• The start symbol of the grammar can have only synthesized attributes.

• In the case of LR parsing with its special start symbol, the start symbol cannot have any inherited attributes because
  1. it does not have any parent nodes in the parse tree and
  2. it does not occur on the right-hand side of any production.
Synthesized Attributes

Evaluating the expression \((4 - 1)/2\) generated by the grammar for subtraction and division
Synthesized Attributes: 0
Synthesized Attributes: 1

Synthesized Attributes
Synthesized Attributes: 2

Synthesized Attributes
Synthesized Attributes: 3
Synthesized Attributes: 4
Synthesized Attributes: 5
Synthesized Attributes: 6
Synthesized Attributes: 7

The diagram shows a decision tree with labeled nodes and branches. The tree has multiple levels, with each node labeled with an attribute (E, T, F) and the values (true, false, unknown) at each branch point.

The synthesized attributes are marked at the bottom of the tree:

- Attribute 4
- Attribute 3
- Attribute 2
- Attribute 1
Synthesized Attributes: 8

![Diagram of synthesized attributes with nodes labeled E, T, F, and numbers indicating attributes.](image)
Synthesized Attributes: 9
Synthesized Attributes: 10

Synthesized Attributes

Diagram showing a tree structure with labeled nodes labeled E, T, F, and numbers indicating node values.
Synthesized Attributes: 11
Synthesized Attributes: 12
Synthesized Attributes: 13
Synthesized Attributes: 14
An Attribute Grammar

\[
\begin{align*}
E_0 & \rightarrow E_1 \cdot T & \triangleright & E_0.val := \text{sub}(E_1.val, T.val) \\
E & \rightarrow T & \triangleright & E.val := T.val \\
T_0 & \rightarrow T_1 \div F & \triangleright & T_0.val := \text{div}(T_1.val, F.val) \\
T & \rightarrow F & \triangleright & T.val := F.val \\
F & \rightarrow (E) & \triangleright & F.val := E.val \\
F & \rightarrow n & \triangleright & F.val := n.val
\end{align*}
\]
Synthesized Attributes Evaluation: Bottom-up

During bottom-up parsing synthesized attributes are evaluated as follows:

**Bottom-up Parsers**

1. Keep an attribute value stack along with the parsing stack.
2. Just before applying a reduction of the form $Z \rightarrow Y_1 \ldots Y_k$ compute the attribute values of $Z$ from the attribute values of $Y_1, \ldots, Y_k$ and place them in the same position on the attribute value stack corresponding to the one where the symbol $Z$ will appear on the parsing stack as a result of the reduction.
Synthesized Attributes Evaluation: Top-down

During top-down parsing synthesized attributes are evaluated as follows:

**Top-down Parsers** In any production of the form \( Z \rightarrow Y_1 \ldots Y_k \), the parser makes recursive calls to procedures corresponding to the symbols \( Y_1 \ldots Y_k \). In each case the attributes of the non-terminal symbols \( Y_1 \ldots Y_k \) are computed and returned to the procedure for \( Z \). Compute the synthesized attributes of \( Z \) from the attribute values returned from the recursive calls.
Inherited Attributes: 0

C-style declarations generating int x, y, z.

\[
D \rightarrow T \, L \\
T \rightarrow \text{int} \mid \text{float} \\
L \rightarrow L, I \mid I \\
I \rightarrow x \mid y \mid z
\]
Inherited Attributes: 1

C-style declarations generating \(\textbf{int} \ x, \ y, \ z.\)

\[
D \rightarrow T \ L \quad T \rightarrow \text{int} \mid \text{float} \\
L \rightarrow L,I \mid I \\
I \rightarrow x \mid y \mid z
\]
Inherited Attributes: 2

C-style declarations generating \textbf{int} \(x, y, z\).

\[
D \rightarrow T \ L \\
T \rightarrow \text{int} \mid \text{float} \\
L \rightarrow L, I \mid I \\
I \rightarrow x \mid y \mid z
\]
Inherited Attributes: 3

C-style declarations generating int x, y, z.

\[
D \rightarrow TL \\
L \rightarrow LI | I \\
T \rightarrow \text{int} | \text{float} \\
I \rightarrow x | y | z
\]
Inherited Attributes: 4

C-style declarations generating `int x, y, z`.

```
D → T L
L → L, I | I
T → int | float
I → x | y | z
```

```
D → T L → L, I | I → x | y | z
I → int
T → int
L → L, I | I → x | y | z
```

```
D
L
T
I
```

```
x y z int
```

```
D L T I
```

```

```
Inherited Attributes: 5

C-style declarations generating int x, y, z.

\[
D \rightarrow TL \\
L \rightarrow LL | I \\
T \rightarrow \text{int} | \text{float} \\
I \rightarrow x | y | z
\]
Inherited Attributes: 6

C-style declarations generating \( \text{int } x, y, z. \)

\[
D \rightarrow T \ L \\
L \rightarrow L, I \mid I \\
T \rightarrow \text{int } \mid \text{float} \\
I \rightarrow x \mid y \mid z
\]
Inherited Attributes: 7

C-style declarations generating \(\text{int } x, y, z\).

\[D \rightarrow T \ L\]
\[L \rightarrow L , I \mid I\]
\[T \rightarrow \text{int } \mid \text{float}\]
\[I \rightarrow x \mid y \mid z\]
Attribute Grammar: Inherited

\[
\begin{align*}
D & \rightarrow TL & \triangleright & L.in := T.type \\
T & \rightarrow \text{int} & \triangleright & T.type := \text{int}.\text{int} \\
T & \rightarrow \text{float} & \triangleright & T.type := \text{float}.\text{float} \\
L_0 & \rightarrow L_1,I & \triangleright & L_1 := L_0.in \\
L & \rightarrow I & \triangleright & I.in := L.in \\
I & \rightarrow \text{id} & \triangleright & \text{id}.\text{type} := I.in
\end{align*}
\]
L-attributed Definitions

Definition 7.4 A grammar is L-attributed if for each production of the form $Y \rightarrow X_1 \ldots X_k$, each inherited attribute of the symbol $X_j$, $1 \leq j \leq k$, depends only on

1. the inherited attributes of the symbol $Y$ and

2. the synthesized or inherited attributes of $X_1, \ldots, X_{j-1}$.
Why L-attributedness?

Intuitively, if $X_{j}.inh$ is an inherited attribute then

- it cannot depend on any synthesized attribute $Y.syn$ of $Y$ because it is possible that the computation of $Y.syn$ requires the value of $X_{j}.inh$ leading to circularity in the definition.

- if the value of $X_{j}.inh$ depends upon the attributes of one or more of the symbols $X_{j+1}, \ldots, X_{k}$ then the computation of $X_{j}.inh$ cannot be performed just before the reduction by the rule $Y \rightarrow X_{1} \ldots X_{k}$ during parsing. Instead it may have to be postponed till the end of parsing.

- it could depend on the synthesized or inherited attributes of any of the symbols $X_{1} \ldots X_{j-1}$ since they would already be available on the attribute value stack.

- it could depend upon the inherited attributes of $Y$ because these inherited attributes can be computed from the attributes of the symbols lying below $X_{1}$ on the stack, provided these inherited attributes of $Y$ are also L-attributed.
A Non L-attributed Definition

Our attribute grammar for C-style declarations is definitely L-attributed. However consider the following grammar for declarations in Pascal and ML.

\[
\begin{align*}
D & \rightarrow L:T \triangleright L.in := T.type \\
T & \rightarrow \text{int} \triangleright T.type := \text{int}.\text{int} \\
T & \rightarrow \text{real} \triangleright T.type := \text{real}.\text{real} \\
L_0 & \rightarrow L_1,I \triangleright L_1 := L_0.in \\
L & \rightarrow I \triangleright I.in := L.in \\
I & \rightarrow \text{id} \triangleright \text{id}.\text{type} := I.in
\end{align*}
\]

In the first semantic rule the symbol \(L.in\) is inherited from a symbol to its right viz. \(T.type\) and hence is not L-attributed.
Evaluating Non-L-attributed Definitions

In many languages like ML which allow higher order functions as values, a definition not being L-attributed may not be of serious concern. But in most other languages it is serious enough to warrant changing the grammar of the language so as to replace inherited attributes by corresponding synthesized ones. The language of the grammar of Pascal and ML declarations can be generated as follows:

\[
D \rightarrow \text{id}L \quad \triangleright \quad \text{addtype}(\text{id}, L.type) \\
L \rightarrow :T \quad \triangleright \quad L.in := T.type \\
L \rightarrow ,\text{id} \; L \quad \triangleright \quad L_0.type := L_1.type; \\
\text{addtype}(\text{id}.L_1.type) \\
T \rightarrow \text{int} \quad \triangleright \quad T.type := \text{int}.int \\
T \rightarrow \text{real} \quad \triangleright \quad T.type := \text{real}.real
\]
Dependency Graphs

In general, the attributes required to be computed during parsing could be synthesized or inherited and further it is possible that some synthesized attributes of some symbols may depend on the inherited attributes of some other symbols. In such a scenario it is necessary to construct a dependency graph of the attributes of each node of the parse tree.
Dependency Graph Construction

**Algorithm 9 Attribute Dependency Graph Construction**

Require: A parse tree of a CFG and the list of attributes

Ensure: A dependency graph

for all nodes $n$ of the parse tree do
  for all attributes $a$ of node $n$ do
    Create an attribute node $n.a$
  end for
end for

for all nodes $n$ of the parse tree do
  for all semantic rules $a ::= f(b_1, \ldots, b_k)$ do
    for all $i : 1 \leq i \leq k$ do
      Create a directed edge $b_i \rightarrow a$
    end for
  end for
end for
8. Abstract Syntax
Abstract Syntax Trees

The construction of ASTs from concrete parse trees is another example of a transformation that can be performed using a syntax-directed definition that has no side-effects. Hence we define it using an attribute grammar.
Abstract Syntax: 0

\[ E \rightarrow E - T \mid T \]
\[ T \rightarrow T / F \mid F \]
\[ F \rightarrow n \mid (E) \]

Suppose we want to evaluate an expression \((4 - 1)/2\). What we actually want is a tree that looks like this:
Evaluation: 0

```
/   2
--
4   1
```
Evaluation: 1

```
 /  \
--- 2
  \
  4 1
```
Evaluation: 2

```
1
/  \\
3   2
```

Evaluation: 2

Evaluation: 3

Diagram: A tree with nodes 1, 2, and 3 connected. Node 1 is the root, node 3 is a child of node 1, and node 2 is another child of node 1.
But what we *actually* get during parsing is a tree that looks like . . .
Abstract Syntax: 1

\ldots THIS!
Abstract Syntax

Shift-reduce parsing produces a concrete syntax tree from the rightmost derivation. The syntax tree is concrete in the sense that

• It contains a lot of redundant symbols that are important or useful only during the parsing stage.
  – punctuation marks
  – brackets of various kinds
• It makes no distinction between operators, operands, and punctuation symbols

On the other hand the abstract syntax tree (AST) contains no punctuations and makes a clear distinction between an operand and an operator.
Abstract Syntax: Imperative Approach

We use attribute grammar rules to construct the abstract syntax tree (AST) from the parse tree. But in order to do that we first require two procedures for tree construction.

**makeLeaf(literal)** : Creates a node with label `literal` and returns a pointer or a reference to it.

**makeBinaryNode(opr, opd1, opd2)** : Creates a node with label `opr` (with fields which point to `opd1` and `opd2`) and returns a pointer or a reference to the newly created node.

Now we may associate a synthesized attribute called `ptr` with each terminal and nonterminal symbol which points to the root of the subtree created for it.
Abstract Syntax Trees: Imperative

\[ E_0 \to E_1 - T \quad \triangleright \quad E_0\.ptr := \text{makeBinaryNode}(\ -, E_1\.ptr, T\.ptr) \]

\[ E \to T \quad \triangleright \quad E\.ptr := T\.ptr \]

\[ T_0 \to T_1 / F \quad \triangleright \quad T_0\.ptr := \text{makeBinaryNode}(/, T_1\.ptr, F\.ptr) \]

\[ T \to F \quad \triangleright \quad T\.ptr := F\.ptr \]

\[ F \to (E) \quad \triangleright \quad F\.ptr := E\.ptr \]

\[ F \to n \quad \triangleright \quad F\.ptr := \text{makeLeaf}(n\.val) \]

The Big Picture
Abstract Syntax: Functional Approach

We use attribute grammar rules to construct the abstract syntax tree (AST) functionally from the parse tree. But in order to do that we first require two functions/constructors for tree construction.

`makeLeaf(literal)` : Creates a node with label `literal` and returns the AST.

`makeBinaryNode(opr, opd1, opd2)` : Creates a tree with root label `opr` (with sub-trees `opd1` and `opd2`).

Now we may associate a synthesized attribute called `ast` with each terminal and nonterminal symbol which points to the root of the subtree created for it.
Abstract Syntax: Functional

\[
E_0 \rightarrow E_1 - T \quad \triangleright \quad E_0.\text{ast} := \text{makeBinaryNode}(-, E_1.\text{ast}, T.\text{ast})
\]

\[
E \rightarrow T \quad \triangleright \quad E.\text{ast} := T.\text{ast}
\]

\[
T_0 \rightarrow T_1 / F \quad \triangleright \quad T_0.\text{ast} := \text{makeBinaryNode}(/, T_1.\text{ast}, F.\text{ast})
\]

\[
T \rightarrow F \quad \triangleright \quad T.\text{ast} := F.\text{ast}
\]

\[
F \rightarrow (E) \quad \triangleright \quad F.\text{ast} := E.\text{ast}
\]

\[
F \rightarrow \text{n} \quad \triangleright \quad F.\text{ast} := \text{makeLeaf}(<n>.\text{val})
\]

The Big Picture
“The name of the song is called ‘Haddock's Eyes’.”

“Oh, that's the name of the song, is it?” Alice said, trying to feel interested.

“No, you don't understand,” the Knight said, looking a little vexed. “That's what the name is called. The name of the song really is, ‘The Aged Aged Man’.”

Then I ought to have said ‘That's what the song is called’?” Alice corrected herself.

“No you oughtn't: that's quite another thing! The song is called ‘Ways and Means’: but that's only what it's called, you know!”

“Well, what is the song, then?” said Alice, who was by this time completely bewildered.

“I was coming to that”, the Knight said. “The song really is ‘A-Sitting On a Gate’: and the tune's my own invention.

Lewis Carroll, *Through the Looking-Glass*
Symbol Table: 1

- The store house of context-sensitive and run-time information about every identifier in the source program.
- All accesses relating to an identifier require to first find the attributes of the identifier from the symbol table.
- Usually organized as a hash table – provides fast access.
- Compiler-generated temporaries may also be stored in the symbol table.
Symbol Table: 2

Attributes stored in a symbol table for each identifier:

- type
- size
- scope/visibility information
- base address
- addresses to location of auxiliary symbol tables (in case of records, procedures, classes)
- address of the location containing the string which actually names the identifier and its length in the string pool
Symbol Table:3

• A symbol table exists throughout the compilation and run-time.
• Major operations required of a symbol table:
  – insertion
  – search
  – deletions are purely logical (depending on scope and visibility) and not physical
• Keywords are often stored in the symbol table before the compilation process begins.
Symbol Table: 4

Accesses to the symbol table at every stage of the compilation process,

**Scanning:** Insertion of new identifiers.

**Parsing:** Access to the symbol table to ensure that an operand exists (declaration before use).

**Semantic analysis:**

- Determination of types of identifiers from declarations
- type checking to ensure that operands are used in type-valid contexts.
- Checking scope, visibility violations.
Symbol Table: 5

**IR generation:** Memory allocation and relative address calculation.

**Optimization:** All memory accesses through symbol table

**Target code:** Translation of relative addresses to absolute addresses in terms of word length, word boundary etc.

*The Big picture*

\[ \text{\textsuperscript{a}i.e.\text{relative}} \text{ to a base address that is known only at run-time} \]
Intermediate Representation
Intermediate Representation

Intermediate representations are important for reasons of portability i.e. platform (hardware and OS) independence.

• (more or less) independent of specific features of the high-level language.
  Example. Java byte-code which is the instruction set of the Java Virtual Machine (JVM).

• (more or less) independent of specific features of any particular target architecture (e.g. number of registers, memory size)
  – number of registers
  – memory size
  – word length
IR Properties: 1

1. It is fairly low-level containing instructions common to all target architectures and assembly languages.
   How low can you stoop? . . .

2. It contains some fairly high-level instructions that are common to most high-level programming languages.
   How high can you rise?

3. To ensure portability
   - an unbounded number of variables and memory locations
   - no commitment to Representational Issues

4. To ensure type-safety
   - memory locations are also typed according to the data they may contain,
   - no commitment is made regarding word boundaries, and the structure of individual data items.

Next
IR: Representation?

• No commitment to word boundaries or byte boundaries
• No commitment to representation of
  – int vs. float,
  – float vs. double,
  – packed vs. unpacked,
  – strings – where and how?.

Back to IR Properties:1
IR: How low can you stoop?

• most arithmetic and logical operations, load and store instructions etc.
• so as to be interpreted easily,
• the interpreter is fairly small,
• execution speeds are high,
• to have fixed length instructions (where each operand position has a specific meaning).

Back to IR Properties:1
IR: How high can you rise?

• typed variables,
• temporary variables instead of registers,
• array-indexing,
• random access to record fields,
• parameter-passing,
• pointers and pointer management
• no limits on memory addresses

Back to IR Properties:1
A typical instruction set: 1

Three address code: A suite of instructions. Each instruction has at most 3 operands.

- an opcode representing an operation with at most 2 operands
- two operands on which the binary operation is performed
- a target operand, which accumulates the result of the (binary) operation.

If an operation requires less than 3 operands then one or more of the operands is made null.
A typical instruction set: 2

• Assignments (LOAD-STORE)
• Jumps (conditional and unconditional)
• Procedures and parameters
• Arrays and array-indexing
• Pointer Referencing and Dereferencing

c.f. Java byte-code
A typical instruction set: 2.1

• Assignments (LOAD-STORE)
  - $x := y \ bop \ z$, where $bop$ is a binary operation
  - $x := uop \ y$, where $uop$ is a unary operation
  - $x := y$, load, store, copy or register transfer

• Jumps (conditional and unconditional)

• Procedures and parameters

• Arrays and array-indexing

• Pointer Referencing and Dereferencing
A typical instruction set: 2.2

- Assignments (LOAD-STORE)
- Jumps (conditional and unconditional)
  - goto L – Unconditional jump,
  - x relop y goto L – Conditional jump, where relop is a relational operator
- Procedures and parameters
- Arrays and array-indexing
- Pointer Referencing and Dereferencing
A typical instruction set: 2.3

- Assignments (LOAD-STORE)
- Jumps (conditional and unconditional)
- Procedures and parameters
  - `call p n`, where `n` is the number of parameters
  - `return y`, return value from a procedures call
  - `param x`, parameter declaration
- Arrays and array-indexing
- Pointer Referencing and Dereferencing
A typical instruction set: 2.4

• Assignments (LOAD-STORE)
• Jumps (conditional and unconditional)
• Procedures and parameters
• Arrays and array-indexing
  – \( x := a[i] \) – array indexing for \( r\)-value
  – \( a[j] := y \) – array indexing for \( l\)-value

Note: The two opcodes are different depending on whether \( l\)-value or \( r\)-value is desired. \( x \) and \( y \) are always simple variables

• Pointer Referencing and Dereferencing
A typical instruction set: 2.5

- Assignments (LOAD-STORE)
- Jumps (conditional and unconditional)
- Procedures and parameters
- Arrays and array-indexing
- Pointer Referencing and Dereferencing
  - $x := \hat{y}$ – referencing: set $x$ to point to $y$
  - $x := \ast y$ – dereferencing: copy contents of location pointed to by $y$ into $x$
  - $\ast x := y$ – dereferencing: copy r-value of $y$ into the location pointed to by $x$

Picture
Pointers

\[ x := ^y \]
\[ *x := y \]
\[ x := *y \]
\[ @z := ^y \]
\[ @z := *y \]
\[ z := *x \]

Diagram:

- Diagram showing pointer assignments and dereferences.
- Variables and pointers are represented with boxes and arrows indicating movement and assignment.
- Illustrates the concept of pointers in computer science.

Note: The diagram is a visual representation of pointer manipulation in a programming context.
IR: Generation Basics

• Can be generated by recursive traversal of the abstract syntax tree.

• Can be generated by syntax-directed translation as follows:
  For every non-terminal symbol $N$ in the grammar of the source language there exist two attributes
  $N$.place, which denotes the address of a temporary variable where the result of the execution of the generated code is stored
  $N$.code, which is the actual code segment generated.

• In addition a global counter for the instructions generated is maintained as part of the generation process.

• It is independent of the source language but can express target machine operations without committing to too much detail.
IR: Infrastructure 1

Given an abstract syntax tree T, with T also denoting its root node.

T.place address of temporary variable where result of execution of the T is stored.

newtemp returns a fresh variable name and also installs it in the symbol table along with relevant information

T.code the actual sequence of instructions generated for the tree T.

newlabel returns a label to mark an instruction in the generated code which may be the target of a jump.

emit emits an instructions (regarded as a string).
IR: Infrastructure 2

Colour and font coding of IR code generation process.

- **Green**: Nodes of the Abstract Syntax Tree
- **Brown**: Intermediate Representation i.e. the language of the “virtual machine”
- **Red**: Variables and data structures of the *language* in which the IR code generator is written
- **Blue**: Names of relevant *procedures* used in IR code generation.
- **Black**: All other stuff.
IR: Expressions

\[ E \rightarrow id \]

\[ E.place := id.place; \]
\[ E.code := emit() \]

\[ E_0 \rightarrow E_1 - E_2 \]

\[ E_0.place := \text{newtemp}; \]
\[ E_0.code := E_1.code; \]
\[ E_2.code; \]
\[ \text{emit}(E_0.place := E_1.place - E_2.place) \]
The WHILE Language

Assume there is a language of expressions (with start symbol $E$) over which the statements are defined. For simplicity assume these are the only constructs of the language.

$$S \to \text{id} := E \quad \text{Assignment}$$
$$| \quad S; S \quad \text{Sequencing}$$
$$| \quad \text{if } E \text{ then } S \text{ else } S \text{fi} \quad \text{Conditional}$$
$$| \quad \text{while } E \text{ do } S \text{ end} \quad \text{Iteration}$$
IR: Assignment and Sequencing

\[ S \rightarrow id := E \]  
\[ S.code := E.code \]
\[ \text{emit}(id.place:=E.place) \]

\[ S_0 \rightarrow S_1; S_2 \]  
\[ S_0.begin := S_1.begin; \]
\[ S_0.after := S_2.after; \]
\[ S_0.code := \text{emit}(S_0.begin:) \]
\[ S_1.code \]
\[ S_2.code \]
\[ \text{emit}(S_0.after:) \]
IR: Conditional

\[ S_0 \rightarrow \text{if } E \text{ then } S_1 \text{ else } S_2 \text{ fi} \]

\[
\begin{align*}
S_0\text{.begin} & := \text{newlabel}; \\
S_0\text{.after} & := S_2\text{.after}; \\
S_0\text{.code} & := \text{emit}(S_0\text{.begin}:) \\
& \quad E\text{.code}; \\
& \quad \text{emit}(\text{if } E\text{.place}= 0 \text{ goto } S_2\text{.begin}); \\
S_1\text{.code} & := \text{emit}(\text{goto } S_0\text{.after}); \\
S_2\text{.code} & := \text{emit}(\text{goto } S_0\text{.after}): \\
\end{align*}
\]
IR: Iteration

\[ S_0 \rightarrow \text{while } E \text{ do } S_1 \text{ end} \]

\[
S_0\.begin := \text{newlabel}; \\
S_0\.after := \text{newlabel}; \\
S_0\.code := \text{emit}(S_0\.begin:) \\
E\.code \\
\text{emit(} \text{if } E\.place = 0 \text{ goto } S_0\.after); \\
S_1\.code; \\
\text{emit(goto } S_0\.begin); \\
\text{emit(} S_0\.after:) \\
\]
IR: Generation End

While generating the intermediate representation, it is sometimes necessary to generate jumps into code that has not been generated as yet (hence the address of the label is unknown). This usually happens while processing

- **forward** jumps
- **short-circuit** evaluation of boolean expressions

It is usual in such circumstances to either fill up the empty label entries in a second pass over the the code or through a process of **backpatching** (which is the maintenance of lists of jumps to the same instruction number), wherein the blank entries are filled in once the sequence number of the target instruction becomes known.
11. The Pure Untyped Lambda Calculus: Basics

Pure Untyped $\lambda$-Calculus: Syntax

The language $\Lambda$ of pure untyped $\lambda$-terms is the smallest set of terms built up from an infinite set $V$ of variables and closed under the following productions

$$L, M, N ::= x \quad \text{Variable}$$

$$\quad | \quad \lambda x[L] \quad \text{Abstraction}$$

$$\quad | \quad (L M) \quad \text{Application}$$

where $x \in V$.

• A Variable denotes a possible binding in the external environment.
• An Abstraction denotes a function which takes a formal parameter.
• An Application denotes the application of a function to an actual parameter.
Free and Bound Variables

Definition 11.1 For any term $N$ the set of free variables and the set of all variables are defined by induction on the structure of terms.

<table>
<thead>
<tr>
<th>$N$</th>
<th>$FV(N)$</th>
<th>$Var(N)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>${x}$</td>
<td>${x}$</td>
</tr>
<tr>
<td>$\lambda x [L]$</td>
<td>$FV(L) - {x}$</td>
<td>$Var(L) \cup {x}$</td>
</tr>
<tr>
<td>$(L M)$</td>
<td>$FV(L) \cup FV(M)$</td>
<td>$Var(L) \cup Var(M)$</td>
</tr>
</tbody>
</table>

- The set of bound variables $BV(N) = Var(N) - FV(N)$.
- The same variable name may be used with different bindings in a single term (e.g. $(\lambda x[x] \lambda x[(x y)])$)
- The brackets “[” and “]” delimit the scope of the bound variable $x$ in the term $\lambda x[L]$.
- $\Lambda_0 \subseteq \Lambda$ is the set of closed $\lambda$-terms (i.e. terms with no free variables).
Notational Conventions

To minimize use of brackets unambiguously

1. $\lambda x_1 x_2 \ldots x_m[L]$ denotes $\lambda x_1[\lambda x_2[\ldots \lambda x_m[L] \ldots]]$ i.e. $L$ is the scope of each of the variables $x_1, x_2, \ldots x_m$.

2. $(L_1 L_2 \ldots L_m)$ denotes $(\ldots (L_1 L_2) \ldots L_m)$ i.e. application is left-associative.
Substitution

Definition 11.2 For any terms \( L, M \) and \( N \) and any variable \( x \), the substitution of the term \( N \) for a variable \( x \) is defined as follows:

\[
\begin{align*}
\{N/x\}x & \equiv N \\
\{N/x\}y & \equiv y \quad \text{if } y \neq x \\
\{N/x\}\lambda x[L] & \equiv \lambda x[L] \\
\{N/x\}\lambda y[L] & \equiv \lambda y[\{N/x\}L] \quad \text{if } y \neq x \text{ and } y \notin FV(N) \\
\{N/x\}\lambda y[L] & \equiv \lambda z[\{N/x\}\{z/y\}L] \quad \text{if } y \neq x \text{ and } y \in FV(N) \text{ and } z \text{ is 'fresh'} \\
\{N/x\}(L M) & \equiv (\{N/x\}L \{N/x\}M)
\end{align*}
\]

- In the above definition it is necessary to ensure that the free variables of \( N \) continue to remain free after substitution i.e. none of the free variables of \( N \) should be "captured" as a result of the substitution.
- The phrase "\( z \) is 'fresh'" may be taken to mean \( z \notin FV(N) \cup Var(L) \).
- \( z \) could be fresh even if \( z \in BV(N) \).
Compatibility

Definition 11.3 A binary relation $\rho \subseteq \Lambda \times \Lambda$ is said to be compatible if $L \rho M$ implies

1. for all variables $x$, $\lambda x[L] \rho \lambda x[M]$ and

2. for all terms $N$, $(L N) \rho (M N)$ and $(N L) \rho (N M)$. 
**Compatible Closure**

**Definition 11.4** The compatible closure of a relation \( \rho \subseteq \Lambda \times \Lambda \) is the smallest (under the \( \subseteq \) ordering) relation \( \rho^c \subseteq \Lambda \times \Lambda \) such that

\[
\begin{align*}
\rho & \quad \frac{L \rho M}{L \rho^c M} \\
\rho \text{Abs} & \quad \frac{L \rho^c M}{\lambda x[L] \rho^c \lambda x[M]} \\
\rho \text{AppL} & \quad \frac{L \rho^c M}{(L N) \rho^c (M N)} \\
\rho \text{AppR} & \quad \frac{L \rho^c M}{(N L) \rho^c (N M)}
\end{align*}
\]

**Lemma 11.5**

1. \( \rho^c \supseteq \rho \).
2. The compatible closure of any relation is compatible.
3. If \( \rho \) is compatible then \( \rho^c = \rho \).

**Example 11.6**

1. \( \equiv_\alpha \) is a compatible relation
2. \( \rightarrow_\beta \) is by definition a compatible relation.
\(\alpha\)-equivalence

**Definition 11.7 (\(\alpha\)-equivalence)**  \(\equiv_{\alpha} \subseteq \Lambda \times \Lambda\) is the *compatible closure of the relation*  
\(\{(\lambda x[L] \equiv_{\alpha} \lambda y[\{y/x\}L]) \mid y \not\in \text{FV}(L)\}\).

- Here again if \(y \in \text{FV}(L)\) it must not be captured by a change of bound variables.
Untyped $\lambda$-Calculus: Basic $\beta$-Reduction

**Definition 11.8**

- Any (sub-)term of the form $(\lambda x[L] M)$ is called a $\beta$-redex.
- *Basic $\beta$-reduction* is the relation on $\Lambda$

$$\rightarrow_\beta \overset{df}{=} \{(\lambda x[L] M), \{M/x\}L') \mid L' \equiv_\alpha L, L', L, M \in \Lambda\}$$

- It is usually represented by the *axiom*

$$\text{(4)} \quad (\lambda x[L] M) \rightarrow_\beta \{M/x\}L'$$

where $L' \equiv_\alpha L$. 
Untyped $\lambda$-Calculus: 1-step $\beta$-Reduction

Definition 11.9 A 1-step $\beta$-reduction $\rightarrow^1_{\beta}$ is the smallest relation (under the $\subseteq$ ordering) on $\Lambda$ such that

\[
\begin{align*}
\beta_1 \quad & L \rightarrow^\beta M \\
& L \rightarrow^1_{\beta} M
\end{align*}
\]

\[
\begin{align*}
\beta_1 \text{Abs} \quad & L \rightarrow^1_{\beta} M \\
& \lambda x[L] \rightarrow^1_{\beta} \lambda x[M]
\end{align*}
\]

\[
\begin{align*}
\beta_1 \text{AppL} \quad & L \rightarrow^1_{\beta} M \\
& (L \ N) \rightarrow^1_{\beta} (M \ N)
\end{align*}
\]

\[
\begin{align*}
\beta_1 \text{AppR} \quad & L \rightarrow^1_{\beta} M \\
& (N \ L) \rightarrow^1_{\beta} (N \ M)
\end{align*}
\]

- $\rightarrow^1_{\beta}$ is the compatible closure of basic $\beta$-reduction to all contexts.

- We will often omit the superscript $^1$ as understood.
Untyped $\lambda$-Calculus: $\beta$-Reduction

Definition 11.10

• For all integers $n \geq 0$, $n$-step $\beta$-reduction $\rightarrow^n_\beta$ is defined by induction on 1-step $\beta$-reduction

\[
\begin{align*}
\text{\textbf{$\beta_n$Basis}} & \quad L \rightarrow^0_\beta L \\
\text{\textbf{$\beta_n$Induction}} & \quad L \rightarrow^m_\beta M \rightarrow^1_\beta N \\
& \quad L \rightarrow^{m+1}_\beta N \quad (m \geq 0)
\end{align*}
\]

• $\beta$-reduction $\rightarrow^*_\beta$ is the reflexive-transitive closure of 1-step $\beta$-reduction. That is,

\[
\begin{align*}
\text{\textbf{$\beta^*$}} & \quad L \rightarrow^n_\beta M \\
& \quad L \rightarrow^*_\beta M \quad (n \geq 0)
\end{align*}
\]
Untyped $\lambda$-Calculus: Normalization

Definition 11.11

- A term is called a $\beta$-normal form ($\beta$-nf) if it has no $\beta$-redexes.
- A term is weakly normalising ($\beta$-WN) if it can reduce to a $\beta$-normal form.
- A term $L$ is strongly normalising ($\beta$-SN) if it has no infinite reduction sequence $L \rightarrow^{1}_\beta L_1 \rightarrow^{1}_\beta L_2 \rightarrow^{1}_\beta \cdots$
Untyped $\lambda$-Calculus: Examples

Example 11.12

1. $K \overset{df}{=} \lambda x \ y[x], \ I \overset{df}{=} \lambda x[x], \ S \overset{df}{=} \lambda x \ y \ z[((x \ z) \ (y \ z))], \ \omega \overset{df}{=} \lambda x[(x \ x)]$ are all $\beta$-nfs.

2. $\Omega \overset{df}{=} (\omega \ \omega)$ has no $\beta$-nf. Hence it is neither weakly nor strongly normalising.

3. $(K \ (\omega \ \omega))$ cannot reduce to any normal form because it has no finite reduction sequences. All its reductions are of the form

$$(K \ (\omega \ \omega)) \rightarrow_{\beta}^{1} (K \ (\omega \ \omega)) \rightarrow_{\beta}^{1} (K \ (\omega \ \omega)) \rightarrow_{\beta}^{1} \cdots$$

or at some point it could transform to

$$(K \ (\omega \ \omega)) \rightarrow_{\beta}^{1} \lambda y[(\omega \ \omega)] \rightarrow_{\beta}^{1} \lambda y[(\omega \ \omega)] \rightarrow_{\beta}^{1} \cdots$$

4. $((K \ \omega) \ \Omega)$ is weakly normalising because it can reduce to the normal form $\omega$ but it is not strongly normalising because it also has an infinite reduction sequence

$$((K \ \omega) \ \Omega) \rightarrow_{\beta}^{1} ((K \ \omega) \ \Omega) \rightarrow_{\beta}^{1} \cdots$$
Examples of Strong Normalization

Example 11.13

1. \(((K \omega) \omega)\) is strongly normalising because it reduces to the normal form \(\omega\) in two \(\beta\)-reduction steps.

2. Consider the term \(((S K) K)\). Its reduction sequences go as follows:

\[
((S K) K) \rightarrow_\beta^1 \lambda z[((K z) (K z))] \rightarrow_\beta^1 \lambda z[z] \equiv I
\]
12. Notions of Reduction

Notions of Reduction
Reduction

For any function such as \( p = \lambda x[3.x.x + 4.x + 1] \),

\[
(p 2) = 3.2.2 + 4.2 + 1 = 21
\]

However there is something asymmetric about the identity, in the sense that while \((p 2)\) deterministically produces \(3.2.2 + 4.2 + 1\) which in turn simplifies deterministically to \(21\), it is not possible to deterministically infer that \(21\) came from \((p 2)\). It would be more accurate to refer to this sequence as a reduction sequence and capture the asymmetry as follows:

\[
(p 2) \rightsquigarrow 3.2.2 + 4.2 + 1 \rightsquigarrow 21
\]

And yet they are behaviourally equivalent and mutually substitutable in all contexts (referentially transparent).

1. Reduction (specifically \(\beta\)-reduction) captures this asymmetry.

2. Since reduction produces behaviourally equal terms we have the following notion of equality.
Untyped $\lambda$-Calculus: $\beta$-Equality

Definition 12.1 $\beta$-equality or $\beta$-conversion (denoted $=_{\beta}$) is the smallest equivalence relation containing $\beta$-reduction ($\rightarrow^{*}_{\beta}$).

The following are equivalent definitions.

1. $=_{\beta}$ is the reflexive-symmetric-transitive closure of 1-step $\beta$-reduction.
2. $=_{\beta}$ is the smallest relation defined by the following rules.

<table>
<thead>
<tr>
<th>$=_{\beta}$ Basis</th>
<th>$L \rightarrow^{*}_{\beta} M$</th>
<th>$L =_{\beta} M$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$=_{\beta}$ Symmetry</td>
<td>$L =_{\beta} M$</td>
<td>$M =_{\beta} L$</td>
</tr>
<tr>
<td>$=_{\beta}$ Reflexivity</td>
<td>$L =_{\beta} L$</td>
<td></td>
</tr>
<tr>
<td>$=_{\beta}$ Transitivity</td>
<td>$L =<em>{\beta} M$, $M =</em>{\beta} N$</td>
<td>$L =_{\beta} N$</td>
</tr>
</tbody>
</table>
The Paradoxical Combinator

Example 12.2 Consider Curry’s paradoxical combinator

\[ Y_C \overset{df}{=} \lambda f[(C \ C)] \]

where

\[ C \overset{df}{=} \lambda x[(f \ (x \ x))] \]

For any term \( L \) we have

\[ (Y_C \ L) \rightarrow^1_\beta (\lambda x[(L \ (x \ x))] \ \lambda x[(L \ (x \ x))]) \]
\[ \equiv^\alpha (\lambda y[(L \ (y \ y))] \ \lambda x[(L \ (x \ x))]) \]
\[ \rightarrow^1_\beta (L \ (\lambda x[(L \ (x \ x))] \ \lambda x[(L \ (x \ x))])) \]
\[ =^\beta (L \ (\ Y_C \ L)) \]

Hence \( (Y_C \ L) =^\beta (L \ (Y_C \ L)) \). However \( (L \ (Y_C \ L)) \) will never \( \beta \)-reduce to \( (Y_C \ L) \).
Recursion and the $Y$ combinator.

Since the lambda calculus only has variables and expressions and there is no place for names themselves (we use names such as $K$ and $S$ for our convenience in discourse, but the language itself allows only (untyped) variables and is meant to define functions anonymously as expressions in the language). In such a situation, recursion poses a problem in the language.

Recursion in most programming languages requires the use of an identifier which names an expression that contains a call to the very name of the function that it is supposed to define. This is at variance with the aim of the lambda calculus wherein the only names belong to variables and even functions may be defined anonymously as mere expressions.

This notion of recursive definitions may be generalised to a system of mutually recursive definitions.

The name of a recursive function, acts as a place holder in the body of the definition (which in turn has the name acting as a place holder for a copy of the body of the definition and so on ad infinitum). However no language can have sentences of infinite length.

The combinator $Y_C$ helps in providing copies of any lambda term $L$ whenever demanded in a more disciplined fashion. This helps in the modelling of recursive definitions anonymously. What the $Y_C$ combinator provides is mechanism for recursion “unfolding” which is precisely our understanding of how recursion should work. Hence it is easy to see from $\left[ (Y_C L) =_\beta \left( L (Y_C L) \right) \right]$ that

\[
(Y_C L) =_\beta \left( L (Y_C L) \right) =_\beta \left( L (L (Y_C L)) \right) =_\beta \left( L (L (L (Y_C L))) \right) =_\beta \cdots
\]

Many other researchers have defined other combinators which mimic the behaviour of the combinator $Y_C$. Of
particular interest is Turing’s combinator \( Y_T \overset{df}{=} (T \; T) \) where \( T \overset{df}{=} \lambda x \; y[(y \; ((x \; x) \; y))] \). Notice that

\[
\begin{align*}
(T \; T) &= (\lambda x \; y[(y \; ((x \; x) \; y))] \; T) \\
\rightarrow^1_\beta &= \lambda y[(y \; ((T \; T) \; y))] \\
&\equiv \lambda y[(y \; (Y_T \; y))]
\end{align*}
\]

from which, by compatible closure, for any term \( L \) we get

\[
\begin{align*}
(Y_T \; L) &= ((A \; A) \; L) \\
\rightarrow^*_\beta &= (\lambda y[(y \; (Y_T \; y))] \; L) \\
\rightarrow^1_\beta &= (L \; (Y_T \; L))
\end{align*}
\]

Thus \( Y_T \) is also a recursion unfolding combinator yielding

\[
(Y_T \; L) =_\beta (L \; (Y_T \; L)) =_\beta (L \; (L \; (Y_T \; L))) =_\beta (L \; (L \; (L \; (Y_T \; L)))) =_\beta \cdots
\]
Compatibility of Beta-reduction and Beta-Equality

Theorem 12.3 $\beta$-reduction $\rightarrow^*_{\beta}$ and $\beta$-equality $=_{\beta}$ are both compatible relations.
Proof of theorem 12.3

Proof: \((\rightarrow^*_{\beta})\) Assume \(L \rightarrow^*_\beta M\). By definition of \(\beta\)-reduction \(L \rightarrow^n_{\beta} M\) for some \(n \geq 0\). The proof proceeds by induction on \(n\)

Basis. \(n = 0\). Then \(L \equiv M\) and there is nothing to prove.

Induction Hypothesis (IH).

The proof holds for all \(k\), \(0 \leq k \leq m\) for some \(m \geq 0\).

Induction Step. For \(n = m + 1\), let \(L \equiv L_0 \rightarrow^m_{\beta} L_m \rightarrow^1_{\beta} M\). Then by the induction hypothesis and the compatibility of \(\rightarrow^1_{\beta}\) we have

\[
\begin{align*}
\text{for all } x \in V, & \quad \lambda x[L] \rightarrow^m_{\beta} \lambda x[L_m], \quad \lambda x[L_m] \rightarrow^1_{\beta} \lambda x[M], & \quad \text{By definition of } \rightarrow^n_{\beta} \\
\text{for all } N \in \Lambda, & \quad (L N) \rightarrow^m_{\beta} (L_m N), \quad (L_m N) \rightarrow^1_{\beta} (M N), & \quad (L N) \rightarrow^n_{\beta} (M N) \\
\text{for all } N \in \Lambda, & \quad (N L) \rightarrow^m_{\beta} (N L_m), \quad (N L_m) \rightarrow^1_{\beta} (N M), & \quad (N L) \rightarrow^n_{\beta} (N M)
\end{align*}
\]

End \((\rightarrow^*_{\beta})\)

\((=_{\beta})\) Assume \(L =_{\beta} M\). We proceed by induction on the length of the proof of \(L =_{\beta} M\) using the definition of \(\beta\)-equality.

Basis. \(n = 1\). Then either \(L \equiv M\) or \(L \rightarrow^*_{\beta} M\). The case of reflexivity is trivial and the case of \(L \rightarrow^*_{\beta} M\) follows from the previous proof.

Induction Hypothesis (IH).
For all terms $L$ and $M$, such that the proof of $L =_{\beta} M$ requires less than $n$ steps for $n > 1$, the compatibility result holds.

**Induction Step.** Suppose the proof requires $n$ steps and the last step is obtained by use of either $=_{\beta} \text{Symmetry}$ or $=_{\beta} \text{Transitivity}$ on some previous steps.

**Case ($=_{\beta} \text{Symmetry}$).** Then the $(n - 1)$-st step proved $M =_{\beta} L$. By the induction hypothesis and then by applying $=_{\beta} \text{Symmetry}$ to each case we get

\[
\begin{align*}
\text{By $=_{\beta} \text{Symmetry}$} \\
\text{for all variables } x, & \quad \lambda x[M] =_{\beta} \lambda x[L] \\
\text{for all terms } N, & \quad (M \, N) =_{\beta} (L \, N) \\
\text{for all terms } N, & \quad (N \, M) =_{\beta} (N \, L)
\end{align*}
\]

**Case ($=_{\beta} \text{Transitivity}$).** Suppose $L =_{\beta} M$ was inferred in the $n$-th step from two previous steps which proved $L =_{\beta} P$ and $P =_{\beta} M$ for some term $P$. Then again by induction hypothesis and then applying $=_{\beta} \text{Transitivity}$ we get

\[
\begin{align*}
\text{By $=_{\beta} \text{Transitivity}$} \\
\text{for all variables } x, & \quad \lambda x[L] =_{\beta} \lambda x[P], \quad \lambda x[P] =_{\beta} \lambda x[M] \\
\text{for all terms } N, & \quad (L \, N) =_{\beta} (P \, N), \quad (P \, N) =_{\beta} (M \, N) \\
\text{for all terms } N, & \quad (N \, L) =_{\beta} (N \, P), \quad (N \, P) =_{\beta} (N \, M)
\end{align*}
\]

End ($=_{\beta}$)

QED
Eta reduction

Given any term $M$ and a variable $x \not\in FV(M)$, the syntax allows us to construct the term $\lambda x[(M \; x)]$ such that for every term $N$ we have

$$(\lambda x[(M \; x)] \; N) \rightarrow^1_\beta (M \; N)$$

In other words,

$$(\lambda x[(M \; x)] \; N) =^\beta (M \; N) \text{ for all terms } N$$

We say that the two terms $\lambda x[(M \; x)]$ and $M$ are extensionally equivalent i.e. they are syntactically distinct but there is no way to distinguish between their behaviours.

So we define basic $\eta$-reduction as the relation

$$\lambda x[(L \; x)] \rightarrow^\eta L \text{ provided } x \not\in FV(L)$$

(5)
Eta-Reduction and Eta-Equality

The following notions are then defined similar to the corresponding notions for $\beta$-reduction.

- **1-step $\eta$-reduction** $\rightarrow^1_\eta$ is the closure of basic $\eta$-reduction to all contexts,
- **$\rightarrow^*_\eta$** is defined by induction on 1-step $\eta$-reduction
- **$\eta$-reduction** $\rightarrow^*\eta$ is the reflexive-transitive closure of 1-step $\eta$-reduction.
- the notions of strong and weak $\eta$ normal forms $\eta$-nf.
- the notion of $\eta$-equality or $\eta$-conversion denoted by $=\eta$. 
Exercise 12.1

1. Prove that $\eta$-reduction and $\eta$-equality are both compatible relations.

2. Prove that $\eta$-reduction is strongly normalising.

3. Define basic $\beta\eta$-reduction as the application of either (4) or (5). Now prove that $\rightarrow_{\beta\eta}$, $\rightarrow^*_{\beta\eta}$ and $=_{\beta\eta}$ are all compatible relations.
Confluence: Definitions
Reduction Relations

Definition 13.1 For any binary relation $\rho$ on $\Lambda$

1. $\rho^1$ is the compatible closure of $\rho$
2. $\rho^+$ is the transitive closure of $\rho^1$
3. $\rho^*$ is the reflexive-transitive-closure of $\rho^1$ and is a preorder
4. $((\rho^1) \cup (\rho^1)^{-1})^*$ (denoted $\equiv_\rho$) is the reflexive-symmetric-transitive closure of $\rho^1$ and is an equivalence relation.
5. $\equiv_\rho$ is also called the equivalence generated by $\rho$.

We will often use $\longrightarrow$ (suitably decorated) as a reduction relation instead of $\rho$. Then $\longrightarrow^1$, $\longrightarrow^+$, $\longrightarrow^*$ and $\longleftarrow^*$ denote respectively the compatible closure, the transitive closure, the reflexive transitive closure and the equivalence generated by $\longrightarrow$
The Diamond Property

Definition 13.2 Let $\rho$ be any relation on terms. $\rho$ has the diamond property if for all $L, M, N$,

$$
\begin{align*}
M & \quad M \\
\rho & \quad \rho \\
L & \Rightarrow \exists P : \\
\rho & \quad \rho \\
N & \quad N
\end{align*}
$$

We often use a decorated version of the symbol $\longrightarrow$ for a reduction relation and depict the diamond property as

$$
\begin{align*}
M \\
\Rightarrow \exists P \\
N
\end{align*}
$$
Reduction Relations: Termination

Let $\rightarrow$ be a reduction relation, $\rightarrow^*$ the least preorder containing $\rightarrow$ and $\leftarrow^*$ the least equivalence relation containing $\rightarrow^*$. Then

**Definition 13.3** $\rightarrow$ is terminating iff there is no infinite sequence of the form

$$L_0 \rightarrow L_1 \rightarrow \cdots$$

**Lemma 13.4** $\rightarrow_\eta$ is a terminating reduction relation.

**Proof:** By induction on the structure of terms. QED
3.1. Why confluence?

We are mostly interested in $\beta$-reduction which is not guaranteed to terminate. We already know that there are several terms which are only weakly normalising ($\beta$-WN). This means that there are several possible reduction sequences, some of which may yield $\beta$-normal forms while the others may yield infinite computations. Hence in order to obtain normal forms for such terms we need to schedule the $\beta$-reductions carefully to be guaranteed a normal form. The matter would be further complicated if there are multiple unrelated normal forms.

Each $\beta$-reduction step may reveal fresh $\beta$-redexes. This in turn raises the disquieting possibility that each termination sequence may yield a different $\beta$-normal form. If such is indeed the case, then it raises fundamental questions on the use of $\beta$-reduction (or function application) as a notion of reduction. If $\beta$-reduction is to be considered fundamental to the notion of computation then all $\beta$-reduction sequences that terminate in $\beta$-nfs must yield the same $\beta$-nf upto $\alpha$-equivalence.

Hence our interest in the notion of confluence. Since the issue of confluence of $\beta$-reduction is rather complicated we approach it in terms of inductively easier notions such as local confluence, and semi-confluence which finally lead up to confluence and the Church-Rosser property.
Reduction: Local Confluence

Definition 13.5 $\rightarrow$ is locally confluent if for all $L, M, N$,

$$N \leftarrow L \rightarrow M \Rightarrow \exists P : N \rightarrow^* P \leftarrow^* M$$

which we denote by

\[
\begin{array}{c}
  M \\
  \uparrow \quad \downarrow^* \\
  L & \Rightarrow \exists P \\
  \downarrow^* \\
  N
\end{array}
\]
Reduction: Semi-confluence

**Definition 13.6** → is semi-confluent if for all \( L, M, N \),

\[
N \leftarrow L \rightarrow^* M \Rightarrow \exists P : N \rightarrow^* P \leftarrow^* M
\]

which we denote by

\[
\begin{array}{ccc}
M & \Rightarrow & \exists P \\
\downarrow & & \downarrow \\
\Rightarrow & & \\
L & \rightarrow^* & \\
\downarrow & & \downarrow \\
N & \rightarrow^* & \\
\end{array}
\]
Reduction: Confluence

Definition 13.7 $\rightarrow$ is confluent if for all $L, M, N$,

$$N \leftarrow L \rightarrow^* M \Rightarrow \exists P : N \rightarrow^* P \leftarrow^* M$$

which we denote as

Fact 13.8 Any confluent relation is also semi-confluent.
Reduction: Church-Rosser

Definition 13.9 \( \rightarrow \) is Church-Rosser if for all \( L, M \),

\[
L \leftrightarrow^* M \Rightarrow \exists P : L \rightarrow^* P \leftarrow^* M
\]

which we denote by

\[
\begin{array}{ccc}
L & \leftrightarrow^* & M \\
* & \downarrow & * \\
\exists P
\end{array}
\]
Equivalence Characterization

Lemma 13.10

1. \( \leftrightarrow^* \) is the least equivalence containing \( \rightarrow \).

2. \( \leftrightarrow^* \) is the least equivalence containing \( \rightarrow^* \).

3. \( L \leftrightarrow^* M \) if and only if there exists a finite sequence \( L \equiv M_0, M_1, \ldots M_m \equiv M, m \geq 0 \) such that for each \( i, 0 \leq i < m \), \( M_i \rightarrow M_{i+1} \) or \( M_{i+1} \rightarrow M_i \). We represent this fact more succinctly as

\[
L \equiv^\alpha M_0 \rightarrow / \leftarrow M_1 \rightarrow / \leftarrow \cdots \rightarrow / \leftarrow M_m \equiv^\alpha M \quad (6)
\]
Proof of lemma 13.10

Proof:

1. Just prove that $\xleftarrow{*} \rightarrow$ is a subset of every equivalence that contains $\rightarrow$.

2. Use induction on the length of proofs to prove this part

3. For the last part it is easy to see that the existence of the “chain equation” (6) implies $L \xleftarrow{*} M$ by transitivity. For the other part use induction on the length of the proof.

QED
The Church-Rosser Property
Confluence and Church-Rosser

Lemma 14.1 Every confluent relation is also semi-confluent

Theorem 14.2 The following statements are equivalent for any reduction relation $\rightarrow$.
1. $\rightarrow$ is Church-Rosser.
2. $\rightarrow$ is confluent.
Proof of theorem 14.2

Proof: (1 ⇒ 2) Assume → is Church-Rosser and let

\[ N \leftrightarrow L \rightarrow^* M \]

Clearly then \( N \leftrightarrow^* M \). If → is Church-Rosser then

\[ \exists P : N \rightarrow^* P \leftrightarrow^* M \]

which implies that it is confluent.

(2 ⇒ 1) Assume → is confluent and let \( L \leftrightarrow^* M \). We proceed by induction on the length of the chain (6).

\[ L \equiv \alpha M_0 \rightarrow / \leftarrow M_1 \rightarrow / \leftarrow \cdots \rightarrow / \leftarrow M_m \equiv \alpha M \]

Basis. \( m = 0 \). This case is trivial since for any \( P, L \rightarrow^* P \iff M \rightarrow^* P \)

Induction Hypothesis (IH).

The claim is true for all chains of length \( k, 0 \leq k < m \).

Induction Step. Assume the chain is of length \( m = k + 1 \). i.e.

\[ L \equiv \alpha M_0 \rightarrow / \leftarrow M_1 \rightarrow / \leftarrow \cdots \rightarrow / \leftarrow M_k \rightarrow / \leftarrow M_{k+1} \equiv \alpha M \]

Case \( M_k \rightarrow M \). Then by the induction hypothesis and semi-confluence we have

\[
\begin{array}{c}
L \\
\downarrow^* \quad \downarrow^* \quad \downarrow^* \quad \downarrow^* \\
\exists Q \\
\exists P \\
M_k \\
M
\end{array}
\]

which proves the claim.
Case $M_k \leftarrow M$. Then the claim follows from the induction hypothesis and the following diagram

\[
\begin{array}{ccc}
L & \leftarrow^* & M_k \leftarrow M \\
\downarrow & \downarrow & \leftarrow^* \\
\exists P
\end{array}
\]

QED

**Lemma 14.3** If a terminating relation is locally confluent then it is semi-confluent.

**Proof**: Assume $L \rightarrow M$ and $L \rightarrow^* N$. We need to show that there exists $P$ such that $M \rightarrow^* P$ and $N \rightarrow^* P$. We prove this by induction on the length of $L \rightarrow^* N$. If $L \equiv_\alpha N$ then $P \equiv_\alpha M$, otherwise assume $L \rightarrow N_1 \rightarrow \cdots \rightarrow N_n = N$ for some $n > 0$. By the local confluence we have there exists $P_1$ such that $M \rightarrow^* P_1$. By successively applying the induction hypothesis we get terms $P_2, \ldots, P_n$ such that $P_{j-1} \rightarrow^* P_j$ and $N_j \rightarrow^* P_j$ for each $j$, $1 \leq j \leq m$. In effect we complete the following rectangle

\[
\begin{array}{ccc}
L & \rightarrow & N_1 \\
\downarrow & \downarrow & \downarrow \\
M & \rightarrow & P_1
\end{array}
\]

QED

From lemma 14.3 and theorem 14.2 we have the following theorem.

**Theorem 14.4** If a terminating relation is locally confluent then it is confluent.

**Proof**: $\rightarrow$ on $\Lambda$ is given to be terminating and locally confluent. We need to show that it is confluent. That is for any $L$, we are given that

1. there is no infinite sequence of reductions of $L$, i.e. every maximal sequence of reductions of $L$ is of length $n$ for some $n \geq 0$. 
2. $N_1 \leftarrow L \rightarrow^1 M_1 \Rightarrow \exists P : M_1 \rightarrow^* P^* \leftarrow N_1$ (7)
We need to show for any term \( L \) that

\[
N \leftarrow L \rightarrow^* M \Rightarrow \exists S : M \rightarrow^* S \leftarrow N
\]

Let \( L \) be any term. Consider the graph \( G(L) = \langle \Gamma(L), \rightarrow^1 \rangle \) such that \( \Gamma(L) = \{ M \mid L \rightarrow^* M \} \). Since \( \rightarrow \) is a terminating reduction

**Fact 14.5** The graph \( G(L) \) is acyclic for any term \( L \).

If \( G(L) \) is not acyclic, there must be a cycle of length \( k > 0 \) such that \( M_0 \rightarrow^1 M_1 \rightarrow^1 \cdots \rightarrow^1 M_{k-1} \rightarrow^1 M_0 \) which implies there is also an infinite reduction sequence of the form \( L \rightarrow^* M_0 \rightarrow^k M_0 \rightarrow^k \cdots \) which is impossible.

Since there are only a finite number of sub-terms of \( L \) that may be reduced under \( \rightarrow \), for each \( L \) there is a maximum number \( p \geq 0 \), which is the length of the longest reduction sequence.

**Fact 14.6** For every \( M \in \Gamma(L) \),

1. \( G(M) \) is a sub-graph of \( G(L) \) and
2. For every \( M \in \Gamma(L) - \{ L \} \), the length of the longest reduction sequence of \( M \) is less than \( p \).

We proceed by induction on \( p \).

**Basis.** \( p = 0 \). Then \( \Gamma(L) = \{ L \} \) and there are no reductions possible, so it is trivially confluent.

**Induction Hypothesis (IH).**

For any \( L \) whose longest reduction sequence is of length \( k \), \( 0 \leq k < p \), property (8) holds.

**Induction Step.** Assume \( L \) is a term whose longest reduction sequence is of length \( p > 0 \). Also assume \( N \leftarrow L \rightarrow^* M \) i.e. \( \exists m, n \geq 0 : N^\dagger \leftarrow L \rightarrow^m M \).

**Case** \( m = 0 \). If \( m = 0 \) then \( M \equiv_\alpha L \) and hence \( S \equiv_\alpha N \).

**Case** \( n = 0 \). Then \( N \equiv_\alpha L \) and we have \( S \equiv_\alpha M \).
Case \( m, n > 0 \). Then consider \( M_1 \) and \( N_1 \) such that
\[
N \leftarrow N_1 \leftarrow L \rightarrow M_1 \rightarrow^* M
\]
See figure (3). By (7), \( \exists P : M_1 \rightarrow^* P \leftarrow N_1 \). Clearly \( M_1, N_1, P \in \Gamma(L) - \{L\} \). Hence by fact 14.6, \( G(M_1), G(N_1) \) and \( G(P) \) are all sub-graphs of \( G(L) \) and all their reduction sequences are of length smaller than \( p \). Hence by induction hypothesis, we get
\[
P \leftarrow M_1 \rightarrow^* M \Rightarrow \exists Q : M \rightarrow^* Q \leftarrow P \tag{10}
\]
and
\[
N \leftarrow N_1 \rightarrow^* P \Rightarrow \exists R : P \rightarrow^* R \leftarrow N \tag{11}
\]
But by (10) and (11) and the induction hypothesis we have
\[
R \leftarrow P \rightarrow^* Q \Rightarrow \exists S : Q \rightarrow^* S \leftarrow R \tag{12}
\]
Combining (12) with (9), (10) and (11) we get
\[
N \leftarrow L \rightarrow^* M \Rightarrow \exists S : M \rightarrow^* S \leftarrow N \tag{13}
\]
QED

**Theorem 14.7** If a terminating relation is locally confluent then it is Church-Rosser.

**Proof:** Follows from theorem 14.4 and theorem 14.2 QED
15. The Church-Rosser Property

The Church-Rosser Property
Parallel Beta Reduction

Definition 15.1 The parallel-$\beta$ or $\parallel\beta$ reduction is the smallest relation for which the following rules hold:

- $\parallel\beta_1$ (L $\rightarrow$ $\parallel\beta$ L)
- $\parallel\beta_1$ Abs1 ($L \rightarrow ^{1}_{\parallel\beta} L'$) ($\lambda x[L] \rightarrow ^{1}_{\parallel\beta} \lambda x[L']$)
- $\parallel\beta_1$ App ($L \rightarrow ^{1}_{\parallel\beta} L'$, $M \rightarrow ^{1}_{\parallel\beta} M'$) ($L M \rightarrow ^{1}_{\parallel\beta} (L' M')$)
- $\parallel\beta_1$ Abs2 ($L \rightarrow ^{1}_{\parallel\beta} L'$, $M \rightarrow ^{1}_{\parallel\beta} M'$) ($(\lambda x[L] M) \rightarrow ^{1}_{\parallel\beta} \{M'/x\}$)
Parallel Beta: The Diamond Property

Lemma 15.2
1. \( L \xrightarrow{\beta} L' \Rightarrow L \xrightarrow{\parallel \beta} L' \).
2. \( L \xrightarrow{\parallel \beta} L' \Rightarrow L \xrightarrow{\parallel \beta} L' \).
3. The smallest preorder containing \( \xrightarrow{\parallel \beta} \) is \( \xrightarrow{\parallel \beta} = \xrightarrow{\parallel \beta} \).
4. If \( L \xrightarrow{\beta} L' \) and \( M \xrightarrow{\parallel \beta} M' \) then \( \{ M/x \} L \xrightarrow{\parallel \beta} \{ M'/x \} L' \).

Proof: By induction on the structure of terms or by induction on the number of steps in any proof. QED

Theorem 15.3 \( \xrightarrow{\parallel \beta} \) has the diamond property.
**Proof of theorem 15.3**

**Proof:** We need to prove for all $L$

\[
N \uparrow^\beta L \rightarrow^\beta M \Rightarrow \exists P : N \rightarrow^\beta P \uparrow^\beta M
\]

We prove this by induction on the structure of $L$ and a case analysis of the rule applied in definition 15.1.

**Case $L \equiv x \in V$.** Then $L \equiv M \equiv N \equiv P$.

Before dealing with the other inductive cases we dispose of some trivial sub-cases that arise in some or all of them.

**Case $L \equiv \alpha M$.** Choose $P \equiv \alpha N$ to complete the diamond.

**Case $L \equiv \alpha N$.** Then choose $P \equiv \alpha M$.

**Case $M \equiv \alpha N$.** Then there is nothing to prove.

In the sequel we assume $N \not\equiv \alpha L \not\equiv \alpha M \not\equiv \alpha N$ and proceed by induction on the structure of $L$.

**Case $L \equiv \lambda x[L_1]$.** Then clearly $M$ and $N$ were both obtained in proofs whose last step was an application of rule $\uparrow^\beta_1 Abs1$ and so $M \equiv \lambda x[M_1]$ and $N \equiv \lambda x[N_1]$ for some $M_1$ and $N_1$ respectively and hence $N_1 \uparrow^\beta \rightarrow^\beta_1 M_1$.

By the induction hypothesis we have

\[
\exists P_1 : N_1 \rightarrow^\beta_1 P_1 \uparrow^\beta M_1
\]

Hence by choosing $P \equiv \lambda x[P_1]$ we obtain the required result.

**Case $L \equiv (L_1 L_2)$ and $L_1$ is not an abstraction.**

The rule $\uparrow^\beta_1 App$ is the only rule that must have been applicable in the last step of the proofs of $N \uparrow^\beta L \rightarrow^\beta M$. Clearly then there exist $M_1$, $M_2$, $N_1$, $N_2$ such that $N_1 \uparrow^\beta \rightarrow^\beta_1 M_1$ and $N_2 \uparrow^\beta \rightarrow^\beta_1 M_2$. Again by the induction hypothesis, we have

\[
\exists P_1 : N_1 \rightarrow^\beta_1 P_1 \uparrow^\beta M_1
\]
and

\[ \exists P_2 : N_2 \rightarrow^1_{\beta} P_2 \rightarrow^1_{\beta} M_2 \]

By choosing \( P \equiv (P_1 P_2) \) we obtain the desired result.

**Case \( L \equiv (\lambda x[L_1] L_2) \).**

Here we have four sub-cases depending upon whether each of \( M \) and \( N \) were obtained by an application of \( ||\beta_1 App \) or \( ||\beta_1 Abs2 \). Of these the sub-case when both \( M \) and \( N \) were obtained by applying \( ||\beta_1 App \) is easy and similar to the previous case. That leaves us with three sub-cases.

**Sub-case:** Both \( M \) and \( N \) were obtained by applying rule \( ||\beta_1 Abs2 \).

Then we have

\[ \{N_2/x\}N_1 \equiv N \rightarrow^1_{\beta} L \equiv (\lambda x[L_1] L_2) \rightarrow^1_{\beta} M \equiv \{M_2/x\}M_1 \]

for some \( M_1, M_2, N_1, N_2 \) such that

\[ N_1 \rightarrow^1_{\beta} L_1 \rightarrow^1_{\beta} M_1 \]

and

\[ N_2 \rightarrow^1_{\beta} L_2 \rightarrow^1_{\beta} M_2 \]

By the induction hypothesis

\[ \exists P_1 : N_1 \rightarrow^1_{\beta} P_1 \rightarrow^1_{\beta} M_1 \]

and

\[ \exists P_2 : N_2 \rightarrow^1_{\beta} P_2 \rightarrow^1_{\beta} M_2 \]

and the last part of lemma 15.2 we have

\[ \exists P \equiv \{P_2/x\}P_1 : N \rightarrow^1_{\beta} P \rightarrow^1_{\beta} M \]
completing the proof.

**Sub-case:** $M$ was obtained by applying rule $\parallel\beta_1\text{Abs2}$ and $N$ by $\parallel\beta_1\text{App}$.

Then we have the form

$$\lambda x [N_1] N_2 \equiv N \quad \text{and} \quad (\lambda x [L_1] L_2) \rightarrow^1 \beta M \equiv \{ M_2/x \} M_1$$

where again

$$N_1 \quad \text{and} \quad N_2 \quad \text{and}$$

$$L_1 \quad \text{and} \quad L_2 \quad \rightarrow^1 \beta M_1 \quad \text{and} \quad \rightarrow^1 \beta M_2$$

By the induction hypothesis

$$\exists P_1 : N_1 \rightarrow^1 \beta P_1 \quad \text{and} \quad \text{by} \quad \beta_1\text{Abs2}$$

$$\exists P_2 : N_2 \rightarrow^1 \beta P_2 \quad \text{and} \quad \text{by} \quad \beta_1\text{App}$$

and finally we have

$$\exists P : N \rightarrow^1 \beta P \quad \text{and} \quad \text{by} \quad \beta_1\text{Abs2}$$

completing the proof.

**Sub-case:** $M$ was obtained by applying rule $\parallel\beta_1\text{App}$ and $N$ by $\parallel\beta_1\text{Abs2}$.

Similar to the previous sub-case.

QED

**Theorem 15.4** $\rightarrow^1 \parallel\beta$ is confluent.
Proof: We need to show that for all \( L, M, N \),

\[
N^{\ast_{\beta}} \rightarrow^{\ast_{\beta}} L \rightarrow^{\ast_{\beta}} M \Rightarrow \exists P : N \rightarrow^{\ast_{\beta}} P \rightarrow^{\ast_{\beta}} M
\]

We prove this by induction on the length of the sequences

\[
L \rightarrow^{1_{\ast_{\beta}}} M_1 \rightarrow^{1_{\ast_{\beta}}} M_2 \rightarrow^{1_{\ast_{\beta}}} \cdots \rightarrow^{1_{\ast_{\beta}}} M_m \equiv M
\]

and

\[
L \rightarrow^{1_{\ast_{\beta}}} N_1 \rightarrow^{1_{\ast_{\beta}}} N_2 \rightarrow^{1_{\ast_{\beta}}} \cdots \rightarrow^{1_{\ast_{\beta}}} N_n \equiv N
\]

where \( m, n \geq 0 \). More specifically we prove this by induction on the pairs of integers \((j, i)\) bounded by \((n, m)\), where \((j, i) < (j', i')\) if and only if either \(j < j'\) or \((j = j')\) and \(i < i'\). The interesting cases are those where both \(m, n > 0\). So we repeatedly apply theorem 15.3 to complete the rectangle

\[
\begin{array}{cccccc}
L & \rightarrow^{1_{\ast_{\beta}}} & M_1 & \rightarrow^{1_{\ast_{\beta}}} & M_2 & \rightarrow^{1_{\ast_{\beta}}} \cdots \rightarrow^{1_{\ast_{\beta}}} M_m \equiv M \\
N_1 & \rightarrow^{1_{\ast_{\beta}}} & P_{11} & \rightarrow^{1_{\ast_{\beta}}} & P_{12} & \rightarrow^{1_{\ast_{\beta}}} \cdots \rightarrow^{1_{\ast_{\beta}}} P_{1m} \\
& \vdots & & \vdots & & \vdots \\
N_n & \rightarrow^{1_{\ast_{\beta}}} & P_{n1} & \rightarrow^{1_{\ast_{\beta}}} & P_{n2} & \rightarrow^{1_{\ast_{\beta}}} \cdots \rightarrow^{1_{\ast_{\beta}}} P_{nm} \equiv P
\end{array}
\]

\(\blacksquare\)

Corollary 15.5 \(\rightarrow^{1_{\ast_{\beta}}}\) is confluent.

Proof: Since \(\rightarrow^{\ast_{\beta}} = \rightarrow^{\ast_{\beta}}\) it follows from theorem 15.4 that \(\rightarrow^{1_{\ast_{\beta}}}\) is confluent. \(\blacksquare\)

Corollary 15.6 If a term reduces to a \(\beta\)-normal form then the normal form is unique (upto \(\equiv_{\alpha}\)).
Proof: If \( N_1 \xrightarrow{\beta} L \xrightarrow{\beta} N_2 \) and both \( N_1, N_2 \) are \( \beta \)-nfs, then by the corollary 15.5 they must both be \( \beta \)-reducible to a third element \( N_3 \) which is impossible if both \( N_1 \) and \( N_2 \) are \( \beta \)-nfs. Hence \( \beta \)-nfs are unique whenever they exist.

Corollary 15.7 \( \xrightarrow{\dagger} \) is Church-Rosser.

Proof: Follows from corollary 15.5 and theorem 14.2.

QED
6.1. FL with recursion

An Applied Lambda-Calculus
A Simple Language of Terms: FL0

Let $X$ be an infinite collection of variables (names). Consider the language (actually a collection of abstract syntax trees) of terms $T_\Omega(X)$ defined by the following constructors (along with their intended meanings).

<table>
<thead>
<tr>
<th>Construct</th>
<th>Arity</th>
<th>Informal Meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>0</td>
<td>The number $0$</td>
</tr>
<tr>
<td>$T$</td>
<td>0</td>
<td>The truth value $true$</td>
</tr>
<tr>
<td>$F$</td>
<td>0</td>
<td>The truth value $false$</td>
</tr>
<tr>
<td>$P$</td>
<td>1</td>
<td>The predecessor function on numbers</td>
</tr>
<tr>
<td>$S$</td>
<td>1</td>
<td>The successor function on numbers</td>
</tr>
<tr>
<td>ITE</td>
<td>3</td>
<td>The if-then-else construct (on numbers and truth values)</td>
</tr>
<tr>
<td>$IZ$</td>
<td>1</td>
<td>The is-zero predicate on numbers</td>
</tr>
<tr>
<td>GTZ</td>
<td>1</td>
<td>The greater-than-zero predicate on numbers</td>
</tr>
</tbody>
</table>
FL(X): Language, Datatype or Instruction Set?

The set of terms $T_\Omega(X)$ may be alternatively defined by the BNF:

$$t ::= x \in X \mid Z \mid (\text{P } t) \mid (\text{S } t) \mid T \mid F \mid (\text{ITE } \langle t, t_1, t_0 \rangle) \mid (\text{IZ } t) \mid (\text{GTZ } t)$$

(14)

• It could be thought of as a user-defined data-type
• It could be thought of as the instruction-set of a particularly simple hardware machine.
• It could be thought of as a simple functional programming language without recursion.
• It is a language with two simple types of data: integers and booleans
• Notice that the constructor $(\text{ITE } \langle t, t_1, t_0 \rangle)$ is overloaded.
Extending the language

To make this simple language safe we require

**Type-checking** : to ensure that arbitrary expressions are not mixed in ways they are not “intended” to be used. For example

- $t$ cannot be a boolean expression in $S(t)$, $P(t)$, $IZ(t)$ and $GTZ(t)$
- $ITE(t, t_1, t_0)$ may be used as a conditional expression for both integers and booleans, but $t$ needs to be a boolean and either both $t_1$ and $t_0$ are integer expressions or both are boolean expressions.

**Functions** : To be a useful programming language we need to be able to define functions.

**Recursion** : to be able to define complex functions in a well-typed fashion. Recursion should also be well-typed
Typing FL Expressions

We have only two types of objects – integers and booleans which we represent by `int` and `bool` respectively. We then have the following elementary typing annotations for the expressions, which may be obtained by pattern matching.

1. \( Z : \text{int} \)
2. \( T : \text{bool} \)
3. \( F : \text{bool} \)
4. \( S : \text{int} \rightarrow \text{int} \)
5. \( P : \text{int} \rightarrow \text{int} \)
6. \( IZ : \text{int} \rightarrow \text{bool} \)
7. \( GTZ : \text{int} \rightarrow \text{bool} \)
8. \( ITEI : \text{bool} \ast \text{int} \ast \text{int} \rightarrow \text{int} \)
9. \( ITEB : \text{bool} \ast \text{bool} \ast \text{bool} \rightarrow \text{bool} \)
Λ+FL(X): The Power of Functions

To make the language powerful we require the ability to define functions, both non-recursive and recursive. We define an applied lambda-calculus of lambda terms $\Lambda_\Omega(X)$ over this set of terms as follows:

$$L, M, N ::= t \in T_\Omega(X) \mid \lambda x[L] \mid (L M) \quad (15)$$

This is two-level grammar combining the term grammar (14) with $\lambda$-abstraction and $\lambda$-application. While this makes it possible to use the operators of $T_\Omega(X)$ as part of functions ($\lambda$-expressions), it does not allow us to use the operators of $T_\Omega(X)$ outside of $\lambda$-abstractions and $\lambda$-applications.
Λ+FL(X): Lack of Higher-order Power?

Example 16.1 The grammar (15) does not allow us to define expressions such as the following:

1. the successor of the result of an application \((S (L M))\) where \((L M)\) yields an integer value.

2. higher order conditionals e.g. \(\lambda x[(ITE \langle (L x), (M x), (N x) \rangle)]\) where \((L x)\) yields a boolean value for an argument of the appropriate type.

3. In general, it does not allow the constructors to be applied to \(\lambda\)-expressions.

So we extend the language by allowing a free intermixing of \(\lambda\)-terms and terms of the sub-language \(T_\Omega(X)\).
\( \Lambda_{FL}(X) \): Higher order functions

We need to \textit{flatten} the grammar of (15) to allow \( \lambda \)-terms also to be used as arguments of the constructors of the term-grammar (14). The language of applied \( \lambda \)-terms (viz. \( \Lambda_{\Omega}(X) \)) now is defined by the grammar.

\[
L, M, N ::= x \in X \mid Z \\
\quad \mid (P \ L) \mid (S \ L) \\
\quad \mid T \mid F \\
\quad \mid (IZ \ L) \mid (GTZ \ L) \\
\quad \mid (ITE \langle L, M, N \rangle) \\
\quad \mid \lambda x[L] \mid (L \ M)
\]

(16)
Unfortunately the result of flattening the grammar leads to an even larger number of meaningless expressions (in particular, we may be able to generate self-referential ones or ones that may not even be interpretable as functions which yield integer or boolean values.

It is therefore imperative that we define a type-checking mechanism to rule out meaningless expressions. As mentioned before, type-checking is not context-free and hence cannot be done through mechanisms such as scanning and parsing and will have to be done separately before any code-generation takes place.

We will in fact, go a step further and design a type-inferencing mechanism that will prevent meaningless expressions from being allowed.

Further, given a well-typed expression we need to be able to define a meaning for each expression that is somehow compatible with our intuitive understanding of what $\lambda$-expressions involving integer and boolean operations mean. This meaning is defined through an operational semantics i.e. a system of transitions on how computation actually takes place for each expression. We define this through a reduction mechanism that is consistent with reduction relations that we have earlier studied for the untyped $\lambda$-calculus.

In order for it to be compatible with the notions of reduction in the $\lambda$-calculus we require to define a notion of reduction first for expressions that do not involve either $\lambda$ abstraction or $\lambda$ application. We refer to this notion of reduction as $\delta$-reduction. Furthermore we need to be able to define $\delta$-normal forms for these expressions. Since the language is completely symbolic, these normal forms would serve as the final answers obtained in the evaluation of these expressions.
The Normal forms for Integers

We need reduction rules to simplify (non-recursive) expressions.

**Zero**. \( Z \) is the unique representation of the number \( 0 \) and every integer expression that is equal to \( 0 \) must be reducible to \( Z \).

**Positive integers**. Each positive integer \( k \) is uniquely represented by the expression \( S^k(Z) \) where the super-script \( k \) denotes a \( k \)-fold application of \( S \).

**Negative integers**. Each negative integer \( -k \) is uniquely represented by the expression \( P^k(Z) \) where the super-script \( k \) denotes a \( k \)-fold application of \( P \).

\( \delta \)-rules

\[
(P (S \ x)) \rightarrow_\delta x
\]

(17)

\[
(S (P \ x)) \rightarrow_\delta x
\]

(18)
Reduction Rules for Boolean Expressions

Pure Boolean Reductions. The constructs $T$ and $F$ are the normal forms for boolean values.

\[
\begin{align*}
(\text{ITE} \langle b, x, x \rangle) & \rightarrow_{\delta} x \\
(\text{ITE} \langle T, x, y \rangle) & \rightarrow_{\delta} x \\
(\text{ITE} \langle F, x, y \rangle) & \rightarrow_{\delta} y
\end{align*}
\]

(19) - (21)

Testing for zero.

\[
\begin{align*}
(\text{IZ Z}) & \rightarrow_{\delta} T \\
(\text{IZ} \langle S \; n \rangle) & \rightarrow_{\delta} F, \text{ where } \langle S \; n \rangle \text{ is a } \delta\text{-nf} \\
(\text{IZ} \langle P \; n \rangle) & \rightarrow_{\delta} F, \text{ where } \langle P \; n \rangle \text{ is a } \delta\text{-nf}
\end{align*}
\]

(22) - (24)
Testing for Positivity

\[
\begin{align*}
\text{(GTZ Z)} & \rightarrow_\delta F & \quad \text{(25)} \\
\text{(GTZ (S n))} & \rightarrow_\delta T, \text{ where } (S n) \text{ is a } \delta\text{-nf} & \quad \text{(26)} \\
\text{(GTZ (P n))} & \rightarrow_\delta F, \text{ where } (P n) \text{ is a } \delta\text{-nf} & \quad \text{(27)}
\end{align*}
\]
Other Non-recursive Operators

We may “program” the other boolean operations as follows:

\[
\text{NOT} \overset{df}{=} \lambda x [\text{ITE} \langle x, F, T \rangle]
\]
\[
\text{AND} \overset{df}{=} \lambda \langle x, y \rangle [\text{ITE} \langle x, y, F \rangle]
\]
\[
\text{OR} \overset{df}{=} \lambda \langle x, y \rangle [\text{ITE} \langle x, T, y \rangle]
\]

We may also “program” the other integer comparison operations as follows:

\[
\text{GEZ} \overset{df}{=} \lambda x [\text{OR} \langle \langle \text{IZ} x \rangle, \langle \text{GTZ} x \rangle \rangle]
\]
\[
\text{LTZ} \overset{df}{=} \lambda x [\text{NOT} \langle \text{GEZ} x \rangle]
\]
\[
\text{LEZ} \overset{df}{=} \lambda x [\text{OR} \langle \langle \text{IZ} x \rangle, \langle \text{LTZ} x \rangle \rangle]
\]
Exercise 16.1

1. Find examples of expressions in FL0 which have more than one computation.

2. Prove that $\rightarrow_\delta$ is terminating.

3. Prove that $\rightarrow_\delta$ is Church-Rosser.

4. Assume we remove the rule 19 from $\delta$-reduction.
   (a) Is it still guaranteed that every boolean expression has a $\delta$-normal form? Prove or find a counter-example.
   (b) Do the $\delta$-normal forms of boolean expressions (whenever they exist) remain the same as before? Prove or find a counter-example.

5. The language FL(X) extends FL0 with variables. What are the new $\delta$-normal forms in FL(X)?
Recursion in the Applied Lambda-calculus

The full power of a programming language will not be realised without a recursion mechanism. The untyped lambda-calculus has “paradoxical combinators” which behave like recursion operators up to $=\beta$.

Definition 16.2 A combinator $Y$ is called a fixed-point combinator if for every lambda term $L$, $(Y\ L) =\beta (L\ (Y\ L))$

Curry’s $Y$ combinator ($Y_C$)

$$Y_C \overset{df}{=} \lambda f[(C\ C)] \quad \text{where} \quad C \overset{df}{=} \lambda x[(f\ (x\ x))]$$

Turing’s $Y$ combinator ($Y_T$)

$$Y_T \overset{df}{=} (T\ T) \quad \text{where} \quad T \overset{df}{=} \lambda y\ x[(x\ (y\ y\ x))]$$

But the various $Y$ combinators unfortunately will not satisfy any typing rules that we may define for the language, because they are all “self-applicative” in nature.
\[ \Lambda_{RecFL}(X) : \text{Adding Recursion} \]

Instead it is more convenient to use the fixed-point property and define a new constructor with a \( \delta \)-rule which satisfies the fixed-point property (definition 16.2).

We extend the language FL with a new constructor

\[ L ::= \ldots \left| (\text{REC } L) \right. \]

and add the fixed point property as a \( \delta \)-rule

\[
(\text{REC } L) \xrightarrow{\delta} (L \; \text{(REC } L)) \quad (28)
\]
Recursion Example: Addition

Consider addition on integers as a binary operation to be defined in this language. We use the following properties of addition on the integers to define it by induction on the first argument.

\[
x + y = \begin{cases} 
  y & \text{if } x = 0 \\
  (x - 1) + (y + 1) & \text{if } x > 0 \\
  (x + 1) + (y - 1) & \text{if } x < 0 
\end{cases}
\]  

(29)
Using the constructors of $\Lambda_{RecFL}(X)$ we require that any (curried) definition of addition on numbers should be a solution to the following equation in $\Lambda_{RecFL}(X)$ for all (integer) expression values of $x$ and $y$.

\[
(plusc \ x \ y) =_{\beta \delta} ITE \langle (IZ \ x), y, ITE \langle (GTZ \ x), (plusc \ (P \ x) \ (S \ y)), (plusc \ (S \ x) \ (P \ y)) \rangle \rangle
\]  

(30)

Equation (30) may be rewritten using abstraction as follows:

\[
plusc =_{\beta \delta} \lambda x \left[ \lambda y \left[ ITE \langle (IZ \ x), y, ITE \langle (GTZ \ x), (plusc \ (P \ x) \ (S \ y)), (plusc \ (S \ x) \ (P \ y)) \rangle \rangle \right] \right]
\]  

(31)

We may think of equation (31) as an equation to be solved in the unknown variable $plusc$.

Consider the (applied) $\lambda$-term obtained from the right-hand-side of equation (31) by simply abstracting the unknown $plusc$.

\[
addc \overset{df}{=} \lambda f \left[ \lambda x \ y \left[ ITE \langle (IZ \ x), y, ITE \langle (GTZ \ x), (f \ (P \ x) \ (S \ y)), (f \ (S \ x) \ (P \ y)) \rangle \rangle \right] \right]
\]  

(32)

Claim 16.3

\[
(REC \ addc) \longrightarrow_{\delta} (addc \ (REC \ addc))
\]  

(33)

and hence

\[
(REC \ addc) =_{\beta \delta} (addc \ (REC \ addc))
\]  

(34)

Claim 16.4 $(REC \ addc)$ satisfies exactly the equation (31). That is

\[
\langle (REC \ addc) \ x \ y \rangle =_{\beta \delta} ITE \langle (IZ \ x), y, ITE \langle (GTZ \ x), ((REC \ addc) \ (P \ x) \ (S \ y)), ((REC \ addc) \ (S \ x) \ (P \ y)) \rangle \rangle
\]  

(35)
Hence we may regard \((\text{REC addc})\) where addc is defined by right-hand-side of definition \((32)\) as the required solution to the equation \((30)\) in which \(plusc\) is an unknown.

The abstraction shown in \((32)\) and the claims \((16.3)\) and \((16.4)\) simply go to show that \(M \equiv_\alpha \lambda f[\{f/z\}L]\) is a solution to the equation \(z =_\beta \delta L\), whenever such a solution does exist. Further, the claims also show that we may “unfold” the recursion (on demand) by simply performing the substitution \(\{L/z\}L\) for each free occurrence of \(z\) within \(L\).
Exercise 16.2

1. Prove that the relation $\rightarrow_\delta$ is confluent.

2. The language FL does not have any operators that take boolean arguments and yields integer values. Define a standard conversion function $B2I$ which maps the value $F$ to $Z$ and $T$ to $(S\ Z)$.

3. Prove that $Y_C$ and $Y_T$ are both fixed-point combinators.

4. Using the combinator $add$ and the other constructs of $\Lambda_\Sigma(X)$ to
   (a) define the equation for products of numbers in the language.
   (b) define the multiplication operation $mult$ on integers and prove that it satisfies the equation(s) for products.

5. The equation (29) is defined conditionally. However the following is equally valid for all integer values $x$ and $y$.

   \[ x + y = (x - 1) + (y + 1) \]  

   (a) Follow the steps used in the construction of $addc$ to define a new applied $addc'$ that instead uses equation (36).
   (b) Is $\beta_\delta (REC\ addc') = \beta_\delta (addc' (REC\ addc'))$?
   (c) Is $addc = \beta_\delta addc'$?
   (d) Is $\beta_\delta (REC\ addc) = \beta_\delta (REC\ addc')$?

6. The function $addc$ was defined in curried form. Use the pairing function in the untyped $\lambda$-calculus, to define
   (a) addition and multiplication as binary functions independently of the existing functions.
   (b) the binary 'curry' function which takes a binary function and its arguments and creates a curried version of the binary function.
6.2. FL with type rules

Typing FL expressions

We have already seen that the simple language FL has

• two kinds of expressions: integer expressions and boolean expressions,

• there are also constructors which take integer expressions as arguments and yield boolean values

• there are also function types which allow various kinds of functions to be defined on boolean expressions and integer expressions.
The Need for typing in FL

• A type is an important *attribute* of any variable, constant or expression, since every such object can only be used in certain kinds of expressions.
• Besides the need for type-checking rules on $T_\Omega(X)$ to prevent illegal constructor operations,
  – rules are necessary to ensure that $\lambda$-applications occur only between terms of appropriate types in order to remain meaningful.
  – rules are necessary to ensure that all terms have clearly defined types at compile-time so that there are no run-time type violations.
TL: A Simple Language of Types

Consider the following language of types (in fully parenthesized form) defined over an infinite collection \( \mathit{TV} \) of type variables, disjoint from the set of variables. We also have two type constants \texttt{int} and \texttt{bool}.

\[
\sigma, \tau ::= \texttt{int} \mid \texttt{bool} \mid \mathit{TV} \mid (\sigma \ast \tau) \mid (\sigma \to \tau)
\]

Notes.

- \texttt{int} and \texttt{bool} are type constants.
- In any type expression \( \tau \), \( TVar(\tau) \) is the set of type variables.
- \( \ast \) is the product operation on types and
- \( \to \) is the function operator on types.
- We require \( \ast \) because of the possibility of defining functions of various kinds of arities in \( \Lambda_X(\Omega) \).
TL: Precedence and Associativity

- **Precedence.** We assume \( \ast \) has a higher precedence than \( \rightarrow \).

- **Associativity.**
  - \( \ast \) is *left associative* whereas
  - \( \rightarrow \) is *right associative*
Type-inference Rules: Infrastructure

The question of assigning types to complicated expressions which may have variables in them still remains to be addressed.

Type inferencing. Can be done using type assignment rules, by a recursive travel of the abstract syntax tree.

Free variables (names) are already present in the environment (symbol table).

Constants and Constructors. May have their types either pre-defined or there may be axioms assigning them types.

Bound variables. May be necessary to introduce “fresh” type variables in the environment.
Type Inferencing: Infrastructure

The elementary typing previously defined for the elementary expressions of FL does not suffice

1. in the presence of $\lambda$ abstraction and application, which allow for higher-order functions to be defined

2. in the presence of polymorphism, especially when we do not want to unnecessarily decorate expressions with their types.
Type Assignment: Infrastructure

• Assume $\Gamma$ is the environment\(^a\) (an association list) which may be looked up to determine the types of individual names. For each variable $x \in X$, $\Gamma(x)$ yields the type of $x$ i.e. $\Gamma(x) = \sigma$ if $x : \sigma \in \Gamma$.

• For each (sub-)expression in FL we define a set $C$ of type constraints of the form $\sigma = \tau$, where $T$ is the set of type variables used in $C$.

• The type constraints are defined by induction on the structure of the expressions in the language FL.

• The expressions of FL could have free variables. The type of the expression would then depend on the types assigned to the free variables. This is a simple kind of polymorphism.

• It may be necessary to generate new type variables as and when required during the process of inferencing and assignment.

\(^a\)usually a part of the symbol table
Constraint Typing Relation

**Definition 16.5** For each term $L \in \Lambda_\Sigma(X)$ the constraint typing relation is of the form

$$\Gamma \vdash L : \tau \triangleright_T C$$

where

- $\Gamma$ is called the context and defines the stack of assumptions that may be needed to assign a type (expression) to the (sub-)expression $L$.
- $\tau$ is the type(-expression) assigned to $L$.
- $C$ is the set of constraints.
- $T$ is the set of “fresh” type variables used in the (sub-)derivations.

---

$^a$usually in the symbol table

$^b$including new type variables
Typing axioms: Basic 1

The following axioms (c.f Typing FL Expressions) may be applied during the scanning and parsing phases of the compiler to assign types to the individual tokens.

<table>
<thead>
<tr>
<th>Axiom</th>
<th>Type Expression</th>
<th>Type Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>Z</td>
<td>$\Gamma \vdash Z : \text{int}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
<tr>
<td>T</td>
<td>$\Gamma \vdash T : \text{bool}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
<tr>
<td>F</td>
<td>$\Gamma \vdash F : \text{bool}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
<tr>
<td>S</td>
<td>$\Gamma \vdash S : \text{int} \rightarrow \text{int}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
<tr>
<td>P</td>
<td>$\Gamma \vdash P : \text{int} \rightarrow \text{int}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
<tr>
<td>IZ</td>
<td>$\Gamma \vdash IZ : \text{int} \rightarrow \text{bool}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
<tr>
<td>GTZ</td>
<td>$\Gamma \vdash GTZ : \text{int} \rightarrow \text{bool}$</td>
<td>$\Delta_\emptyset \emptyset$</td>
</tr>
</tbody>
</table>
Typing axioms: Basic 2

\[
\begin{array}{c}
\text{ITEI} \\
\Gamma \vdash \text{ITE} : \text{bool} \times \text{int} \times \text{int} \to \text{int} \\
\end{array}
\]

\[
\begin{array}{c}
\top \top \top \\
\end{array}
\]

\[
\begin{array}{c}
\text{ITEB} \\
\Gamma \vdash \text{ITE} : \text{bool} \times \text{bool} \times \text{bool} \to \text{bool} \\
\end{array}
\]

\[
\begin{array}{c}
\top \top \top \\
\end{array}
\]

Notice that the constructor \text{ITE} is \textit{overloaded} and actually is two constructors \text{ITEI} and \text{ITEB}. Which constructor is actually used will depend on the context and the type-inferring mechanism.
Type Rules for FL: 3

\[
\text{Var} \quad \frac{}{\Gamma \vdash x : \Gamma(x) \triangleright \emptyset \emptyset}
\]

\[
\text{Abs} \quad \frac{\Gamma, x : \sigma \vdash L : \tau \triangleright_T C}{\Gamma \vdash \lambda x[L] : \sigma \rightarrow \tau \triangleright_T C}
\]

\[
\text{App} \quad \frac{\Gamma \vdash L : \sigma \triangleright_{T_1} C_1 \quad \Gamma \vdash M : \tau \triangleright_{T_2} C_2}{\Gamma \vdash (L M) : \text{'a} \triangleright_{T'} C'}
\]

(Conditions 1. and 2.)

where

- **Condition 1.** \(T_1 \cap T_2 = T_1 \cap TVar(\tau) = T_2 \cap TVar(\sigma) = \emptyset\)

- **Condition 2.** \text{'a} \notin T_1 \cup T_2 \cup TVar(\sigma) \cup TVar(\tau) \cup TVar(C_1) \cup TVar(C_2)\)

- \(T' = T_1 \cup T_2 \cup \{\text{'a}\}\)

- \(C' = C_1 \cup C_2 \cup \{\sigma = \tau \rightarrow \text{'a}\}\)
Example 16.6 Consider the following simple combinator \( \lambda x[\lambda y[\lambda z[(x (y z))]]] \) which defines the function composition operator. Since there are three bound variables \( x, y \) and \( z \) we begin with an initial assumption \( \Gamma = x : 'a, y : 'b, z : 'c \) which assign arbitrary types to the bound variables, represented by the type variables 'a, 'b and 'c respectively. Note however, that since it has no free variables, its type does not depend on the types of any variables. We expect that at the end of the proof there would be no assumptions. Our inference for the type of the combinator then proceeds as follows.

1. \( x : 'a, y : 'b, z : 'c \vdash x : 'a \triangleright_\emptyset \emptyset \)  
   (Var)
2. \( x : 'a, y : 'b, z : 'c \vdash y : 'b \triangleright_\emptyset \emptyset \)  
   (Var)
3. \( x : 'a, y : 'b, z : 'c \vdash z : 'c \triangleright_\emptyset \emptyset \)  
   (Var)
4. \( x : 'a, y : 'b, z : 'c \vdash (y z) : 'd \triangleright \{ 'd \} \{ 'b = 'c \rightarrow 'd \} \)  
   (App)
5. \( x : 'a, y : 'b, z : 'c \vdash (x (y z)) : 'e \triangleright \{ 'd, 'e \} \{ 'b = 'c \rightarrow 'd, 'a = 'd \rightarrow 'e \} \)  
   (App)
6. \( x : 'a, y : 'b \vdash \lambda z[(x (y z))] : 'c \rightarrow 'e \triangleright \{ 'd, 'e \} \{ 'b = 'c \rightarrow 'd, 'a = 'd \rightarrow 'e \} \)  
   (Abs)
7. \( x : 'a \vdash \lambda x[\lambda y[\lambda z[(x (y z))]]] : 'b \rightarrow 'c \rightarrow 'e \triangleright \{ 'd, 'e \} \{ 'b = 'c \rightarrow 'd, 'a = 'd \rightarrow 'e \} \)  
   (Abs)
8. \( \vdash \lambda x[\lambda y[\lambda z[(x (y z))]]] : 'a \rightarrow 'b \rightarrow 'c \rightarrow 'e \triangleright \{ 'd, 'e \} \{ 'b = 'c \rightarrow 'd, 'a = 'd \rightarrow 'e \} \)  
   (Abs)

Hence \( \lambda x[\lambda y[\lambda z[(x (y z))]]] : 'a \rightarrow 'b \rightarrow 'c \rightarrow 'e \) subject to the constraints given by \( \{ 'b = 'c \rightarrow 'd, 'a = 'd \rightarrow 'e \} \) which yields \( \lambda x[\lambda y[\lambda z[(x (y z))]]] : ('d \rightarrow 'e) \rightarrow ('c \rightarrow 'd) \rightarrow 'c \rightarrow 'e \)
Principal Type Schemes

Definition 16.7 A solution for $\Gamma \vdash L : \tau \triangleright_T C$ is a pair $\langle S, \sigma \rangle$ where $S$ is a substitution of type variables in $\tau$ such that $S(\tau) = \sigma$.

- The rules yield a principal type scheme for each well-typed applied $\lambda$-term.
- The term is ill-typed if there is no solution that satisfies the constraints.
- Any substitution of the type variables which satisfies the constraints $C$ is an instance of the most general polymorphic type that may be assigned to the term.
Exercise 16.3

1. The language has several constructors which behave like functions. Derive the following rules for terms in $T_\Omega(X)$ from the basic typing axioms and the rule App.

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Sx}$</td>
<td>$\frac{\Gamma \vdash t : \tau \triangleright_T C}{\Gamma \vdash (S , t) : \text{int} \triangleright_T C \cup {\tau = \text{int}}}$</td>
</tr>
<tr>
<td>$\text{Px}$</td>
<td>$\frac{\Gamma \vdash t : \tau \triangleright_T C}{\Gamma \vdash (P , t) : \text{int} \triangleright_T C \cup {\tau = \text{int}}}$</td>
</tr>
<tr>
<td>$\text{IZx}$</td>
<td>$\frac{\Gamma \vdash t : \tau \triangleright_T C}{\Gamma \vdash (IZ , t) : \text{bool} \triangleright_T C \cup {\tau = \text{int}}}$</td>
</tr>
<tr>
<td>$\text{GTZx}$</td>
<td>$\frac{\Gamma \vdash t : \tau \triangleright_T C}{\Gamma \vdash (GTZ , t) : \text{bool} \triangleright_T C \cup {\tau = \text{int}}}$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Rule</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{ITEx}$</td>
<td>$\frac{\Gamma \vdash t : \sigma \triangleright_T C'}{\Gamma \vdash t_1 : \tau_1 \triangleright_{T_1} C_1 \quad (T \cap T_1 = T_1 \cap T_0 = T_0 \cap T = \emptyset)}$</td>
</tr>
</tbody>
</table>

where $T' = T \cup T_1 \cup T_0$ and $C' = C \cup C_1 \cup C_0 \cup \{\sigma = \text{bool}, \tau = \text{v}\}$
2. Use the rules to define the type of the combinators $K$ and $S$?

3. How would you define a type assignment for the recursive function $\text{addc}$ defined by equation (32)?

4. Prove that the terms, $\omega = \lambda x[(x \ x)]$ and $\Omega = (\omega \ \omega)$ are ill-typed.

5. Are the following well-typed or ill-typed? Prove your answer.

   (a) $(K \ S)$
   (b) $((K \ S) \ \omega)$
   (c) $(((S \ K) \ K) \ \omega)$
   (d) $(\text{ITE} \ ((\text{IZ} \ x), \ T, (K \ x)))$
17. An Imperative Language

An Imperative Language
The Concept of State

• Any imperative language indirectly exposes the *memory* (or *store*) to the user for manipulation.
• Memory (or store) is a set $Loc$ of locations used to store the values of variables.
• We define the store to be a (partial) function from $Loc$ to values. $\sigma : Loc \rightarrow (\text{int} \cup \text{bool})$. $Stores = \{ \sigma \mid \sigma : Loc \rightarrow (\text{int} \cup \text{bool}) \}$ is the set of possible stores.
• Each variable in an imperative program is assigned a location.
• The *environment* is an association $\gamma$ of variable (names) to locations i.e. $\gamma : X \rightarrow Loc$.
• The *(dynamic)* *state* of a program is defined by the pair $(\gamma, \sigma)$.
State: Illustration

• l-values. \( \gamma(x) = i : \text{bool}, \gamma(y) = j : \text{int}, \gamma(z) = k : \text{int} \)

• r-values. \( \sigma(i) = T : \text{bool}, \sigma(j) = 132456 : \text{int}, \sigma(k) = -87567 : \text{int} \)
References in Languages

**ML-like** impure functional languages

- have an explicit polymorphic `'a ref` type constructor. Hence \( x : \text{bool ref}, y, z : \text{int ref} \) and \( x \) is a named reference to the location \( i \)
- have an explicit unary dereferencing operator `!` to read the value contained in the location referenced by \( x \), i.e. \( !x = \sigma(i) \).
- The actual locations however are not directly visible.

**C-like** imperative languages are not as fussy as the ML-like languages. C (and C++) even treats locations only as integers and allows integer operations to be performed on them!
l-values and r-values

• \( l \) is the l-value of \( w \) i.e. \( \gamma(w) = l \in Loc \)
• \( m \) is the r-value of \( w \) i.e. \( \sigma(\gamma(w)) = !w = m \in Loc \)
• \( m \) is also an l-value since \( !w : \text{int ref} \)
• \( !(\text{int ref} w) = 78663 : \text{int} \) is the r-value of \( !w \)
17.1.1. l-values, r-values, aliasing and indirect addressing

The terms “l-value” (for “left-value”) and “r-value” (for “right-value”) come from the practice in most imperative languages of writing assignment commands by overloading the variable name to denote both its address ($\gamma(x)$) in $Loc$ as well as the value $\sigma(\gamma(x))$ stored in memory. Consider the example,

- $x := x + y$ (Pascal)
- $x = x + y$ (C, C++, Java, Python, Perl)

The occurrence of “$x$” on the left-hand side of the assignment command denotes a location $\gamma(x)$ whereas the occurrences of “$x$” and “$y$” on the right-hand-side of the assignment denote the values $\sigma(\gamma(x))$ and $\sigma(\gamma(y))$ respectively. The term “dereferencing” is used to denote the action of “reading” the value stored in a location.

- This notation for assignment becomes a source of tremendous confusion when locations are also valid values, as in the case of indirect addressing (look at $w$) and may be manipulated.
- The confusion is further exacerbated when locations are also integers indistinguishable from the integers stored in the locations. The result of dereferencing an integer variable may be one of the following.
  - An invalid location leading to a segmentation fault. For instance, the integer could be negative or larger than any valid memory address.
  - Another valid location with an undefined value or with a value defined previously when the location was assigned to some other variable in a different job. This could lead to puzzling results in the current program.
  - Another valid location which is already the address of a variable in the program (leading to an aliasing totally unintended by the programmer). This could also lead to puzzling results in the current program.
- Modern impure functional languages (which have strong-typing facilities) usually clearly distinguish between locations and values as different types. Hence every imperative variable represents only an l-value. Its r-value is
obtained by applying a dereferencing operation (the prefix operation \( 1 \)). Hence the same assignment command in ML-like languages would be written

\[- x :=!x+y \] (ML and OCaml)

The following interactive ML session illustrates aliasing and the effect on the aliased variables.

```ml
Standard ML of New Jersey v110.76 [built: Tue Oct 22 14:04:11 2013]
- val u = ref 1;
 val u = ref 1 : int ref
- val v = u; (* u and v are aliases for the same location *)
 val v = ref 1 : int ref
- v := !v+1;
 val it = () : unit
- !u;
 val it = 2 : int
- !v;
 val it = 2 : int
- v := !v+1;
 val it = () : unit
- !u;.val it = 3 : int
- !v;
 val it = 3 : int
```

The following ML-session illustrates indirect addressing (and if you get confused, don’t come to me, I am confused
too; confusion is the price we pay for indiscriminate modification of state).

Standard ML of New Jersey v110.76 [built: Tue Oct 22 14:04:11 2013]
- val x = ref (ref 0);
val x = ref (ref 0) : int ref ref
- val y = !x;
val y = ref 0 : int ref
- val z = ref y;
val z = ref (ref 0) : int ref ref
- y := !y+1;
val it = () : unit
- !y;
val it = 1 : int
- !z;
val it = ref 1 : int ref
- !(!z);
val it = 1 : int
- !(!x);
val it = 1 : int
-
Operational Semantics of Expressions

• Consider the language of terms defined FL0. Instead of the $\delta$-rules defined earlier, we assume that these terms are evaluated on a hardware which can represent int and bool.

• Assume int is the hardware representation of the integers and bool $= \{T, F\}$.

• We assume that every (sub-)expression in the language has been typed with a unique type attribute.

• We define an expression evaluation relation $\rightarrow_e$ such that

$$\rightarrow_e \subseteq (\text{Stores} \times T_\Sigma(X)) \times (\text{Stores} \times (T_\Sigma(X) \cup \text{int} \cup \text{bool}))$$

in a given environment $\gamma$. 
7.2. The Operational Semantics of Commands

WHILE: Big-Step Semantics
The WHILE language

• We initially define a simple language of commands.
• The expressions of the language are those of any term algebra $T_{\Sigma}(X)$.
• We simply assume there is a well-defined relation $\rightarrow_e$ for evaluating expressions in
• We defer defining the relation $\rightarrow_e$ for FL0.
State Changes or Side-Effects

- State changes are usually programmed by assignment commands which occur one location at a time.

- In the simple WHILE language side-effects do not occur except by explicit assignment commands.
Modelling a Side-Effect

Given a store $\sigma$, a variable $x$ such that $\gamma(x) = \ell$ and $\sigma(\ell) = a$, the state change effected by the assignment $x := b$ is a new store that is identical to $\sigma$ except at the location $\gamma(x)$ which now contains the value $b$

$$\sigma' = [\gamma(x) \mapsto b] \sigma$$

i.e.

$$\sigma'(\ell) = \begin{cases} 
\sigma(\ell) & \text{if } \ell \neq \gamma(x) \\
 b & \text{otherwise}
\end{cases}$$
**Aliases**

**Definition 17.1** Two (or more) variables are called **aliases** if they denote the same location ($y$ and $u$ in the figure below).

![Diagram showing variables i, j, k, l, m, x, y, u, z, w with arrows connecting them, indicating alias relationships.](image-url)
The Commands of the WHILE Language

\[ c_0, c_1, c ::= \text{skip} \quad \text{Skip} \]
\[ x ::= e \quad \text{Assign} \]
\[ \{c_0\} \quad \text{Block} \]
\[ c_0; c_1 \quad \text{Seq} \]
\[ \text{if } e \text{ then } c_1 \text{ else } c_0 \quad \text{Cond} \]
\[ \text{while } e \text{ do } c \quad \text{While} \]

where \( e \) is either an integer or boolean expression in the language FL with operational semantics as given before.
Operational Semantics: Basic Commands

**Skip**

\[
\gamma \vdash \langle \sigma, \text{skip} \rangle \rightarrow^1_c \sigma
\]

**Assign**

\[
\gamma \vdash \langle \sigma, e \rangle \rightarrow e \ m
\]

\[
\gamma \vdash \langle \sigma, x := e \rangle \rightarrow^1_c [\gamma(x) \mapsto m] \sigma
\]

**Notes:**

1. The **Skip** rule corresponds to any of the following:
   - a noop
   - the identity function or identity relation on states
   - a command which has no effect on states

2. The assignment is the only command in our language which creates a side-effect (actually changes state)
Operational Semantics: Blocks

We have defined a block as simply a command enclosed in braces. It is meant to delimit a (new) scope. Later we will see that there could be local declarations as well, in which case the semantics changes slightly to include a new scope.

\[
\begin{align*}
\text{Block} & \\
\gamma \vdash \langle \sigma, c \rangle & \xrightarrow{\Gamma_c} \sigma' \\
\gamma \vdash \langle \sigma, \{c\} \rangle & \xrightarrow{\Gamma_c} \sigma'
\end{align*}
\]
Operational Semantics: Sequencing

\[
\begin{array}{c}
\gamma \vdash \langle \sigma, c_0 \rangle \xrightarrow{\gamma_c} \sigma', \\
\gamma \vdash \langle \sigma', c_1 \rangle \xrightarrow{\gamma_c} \sigma'' \\
\gamma \vdash \langle \sigma, c_0; c_1 \rangle \xrightarrow{\gamma_c} \sigma''
\end{array}
\]

Notice that sequencing is precisely the composition of relations. If the relation $\xrightarrow{\gamma_c}$ is a function (in the case of our language it actually is a function), sequencing would then be a composition of functions.
Operational Semantics: Conditionals

\[
\begin{align*}
\text{Cond0} & : & \gamma \vdash \langle \sigma, e \rangle & \rightarrow_e F, \\
& & \gamma \vdash \langle \sigma, c_0 \rangle & \rightarrow^1_c \sigma_0 \\
\hline
& \gamma \vdash \langle \sigma, \text{if } e \text{ then } c_1 \text{ else } c_0 \rangle & \rightarrow^1_c \sigma_0 \\
\end{align*}
\]

\[
\begin{align*}
\text{Cond1} & : & \gamma \vdash \langle \sigma, e \rangle & \rightarrow_e T, \\
& & \gamma \vdash \langle \sigma, c_1 \rangle & \rightarrow^1_c \sigma_1 \\
\hline
& \gamma \vdash \langle \sigma, \text{if } e \text{ then } c_1 \text{ else } c_1 \rangle & \rightarrow^1_c \sigma_1 \\
\end{align*}
\]
Operational Semantics: While loop

We use the fact that the `while e do c` is really a form of recursion – actually it is a form of “tail recursion”. Hence the execution behaviour of `while e do c` is exactly that of

```
if e then {c; while e do c} else skip
```

(37)

The following rules may be derived from (37) using the rules for conditional, sequencing and skip (though the number of steps may not exactly correspond).

**While0**

\[ \Gamma \vdash \langle \sigma, e \rangle \xrightarrow{e} F \]

\[ \Gamma \vdash \langle \sigma, \text{while } e \text{ do } c \rangle \xrightarrow{1\ c} \sigma \]

**While1**

\[ \Gamma \vdash \langle \sigma, e \rangle \xrightarrow{e} T, \]

\[ \Gamma \vdash \langle \sigma, c \rangle \xrightarrow{1\ c} \sigma', \]

\[ \Gamma \vdash \langle \sigma', \text{while } e \text{ do } c \rangle \xrightarrow{1\ c} \sigma'' \]

\[ \Gamma \vdash \langle \sigma, \text{while } e \text{ do } c \rangle \xrightarrow{1\ c} \sigma'' \]
7.3. The Semantics of Expressions in FL

Operational Semantics for FL
Evaluating FL on a machine

• We previously treated FL as simply a data-type and gave $\delta$-rules.
• Here we define a deterministic evaluation mechanism $\rightarrow^e$ on a more realistic hardware which supports integers and booleans.
• The normal forms on this machine would have to be appropriate integer and boolean constants as represented in the machine.
Operational Semantics: Constants and Variables

Let $\sigma \in \text{States}$ be any state.

$$
\begin{align*}
T & \quad \gamma \vdash \langle \sigma, T \rangle \rightarrow_e \langle \sigma, T \rangle \\
F & \quad \gamma \vdash \langle \sigma, F \rangle \rightarrow_e \langle \sigma, F \rangle \\
Z & \quad \gamma \vdash \langle \sigma, Z \rangle \rightarrow_e \langle \sigma, 0 \rangle \\
X & \quad \gamma \vdash \langle \sigma, x \rangle \rightarrow_e \langle \sigma, \sigma(\gamma(x)) \rangle
\end{align*}
$$
Operational Semantics: Integer Expressions

\[
\begin{align*}
\text{P} & \quad \gamma \vdash \langle \sigma, e \rangle \rightarrow e \langle \sigma, m \rangle \quad (e, m : \text{int}) \\
& \quad \gamma \vdash \langle \sigma, (P e) \rangle \rightarrow e \langle \sigma, m - 1 \rangle \\
\text{S} & \quad \gamma \vdash \langle \sigma, e \rangle \rightarrow e \langle \sigma, m \rangle \quad (e, m : \text{int}) \\
& \quad \gamma \vdash \langle \sigma, (P e) \rangle \rightarrow e \langle \sigma, m + 1 \rangle \\
\text{IZ0} & \quad \gamma \vdash \langle \sigma, e \rangle \rightarrow e \langle \sigma, m \rangle \\
& \quad \gamma \vdash \langle \sigma, (IZ e) \rangle \rightarrow e \langle \sigma, F \rangle \quad (e, m : \text{int}, m \not= 0) \\
\text{IZ1} & \quad \gamma \vdash \langle \sigma, e \rangle \rightarrow e \langle \sigma, 0 \rangle \\
& \quad \gamma \vdash \langle \sigma, (IZ e) \rangle \rightarrow e \langle \sigma, T \rangle \quad (e : \text{int}) \\
\text{GTZ0} & \quad \gamma \vdash \langle \sigma, e \rangle \rightarrow e \langle \sigma, m \rangle \\
& \quad \gamma \vdash \langle \sigma, (GTZ e) \rangle \rightarrow e \langle \sigma, F \rangle \quad (e, m : \text{int}, m \leq 0) \\
\text{GTZ1} & \quad \gamma \vdash \langle \sigma, e \rangle \rightarrow e \langle \sigma, m \rangle \\
& \quad \gamma \vdash \langle \sigma, (GTZ e) \rangle \rightarrow e \langle \sigma, T \rangle \quad (e, m : \text{int}, m > 0)
\end{align*}
\]
### Operational Semantics: Conditional Expressions

**ITEI0**
\[
\frac{\gamma \vdash \langle \sigma, e \rangle \rightarrow_e \langle \sigma, F \rangle}{\gamma \vdash \langle \sigma, (ITE \langle e, e_1, e_0 \rangle) \rangle \rightarrow_e \langle \sigma, e_0 \rangle} \quad (e_1, e_0 : \text{int})
\]

**ITEI1**
\[
\frac{\gamma \vdash \langle \sigma, e \rangle \rightarrow_e \langle \sigma, T \rangle}{\gamma \vdash \langle \sigma, (ITE \langle e, e_1, e_0 \rangle) \rangle \rightarrow_e \langle \sigma, e_1 \rangle} \quad (e_1, e_0 : \text{int})
\]

**ITEB0**
\[
\frac{\gamma \vdash \langle \sigma, e \rangle \rightarrow_e \langle \sigma, F \rangle}{\gamma \vdash \langle \sigma, (ITE \langle e, e_1, e_0 \rangle) \rangle \rightarrow_e \langle \sigma, e_0 \rangle} \quad (e_1, e_0 : \text{bool})
\]

**ITEB1**
\[
\frac{\gamma \vdash \langle \sigma, e \rangle \rightarrow_e \langle \sigma, T \rangle}{\gamma \vdash \langle \sigma, (ITE \langle e, e_1, e_0 \rangle) \rangle \rightarrow_e \langle \sigma, e_1 \rangle} \quad (e_1, e_0 : \text{bool})
\]
7.4. The Operational Semantics of Declarations

Local Declarations

We introduce declarations through a new syntactic category $Decls$ defined as follows:

$$d_1, d_2, d ::= \text{int } x \mid \text{bool } y \mid d_1; d_2$$

$$c ::= \cdots \mid \{d; c\}$$

- Most languages insist on a “declaration before use” discipline,
- Declarations create “little new environments”.
- Need to be careful about whether a variable is at all defined.
- Even if the l-value of a variable is defined, its r-value may not be defined. The rules for variables and assignments then need to be changed to the following.
Some changed rules

- We use the symbol $\bot$ to denote the undefined.
- We use $z \neq \bot$ to denote that $z$ is well-defined.

$x' \quad \frac{\gamma \vdash \langle \sigma, x \rangle \rightarrow_e \langle \sigma, \sigma(\gamma(x)) \rangle \quad (\sigma(\gamma(x)) \neq \bot)}{\gamma \vdash \langle \sigma, x \rangle -\rightarrow \langle \sigma, \sigma(\gamma(x)) \rangle}$

\[ \text{Assgn0'} \quad \frac{\gamma \vdash \langle \sigma, x := m \rangle \rightarrow_1 \sigma \quad (\gamma(x) \neq \bot)}{\gamma \vdash \langle \sigma, x \rangle \rightarrow \sigma(x) \mapsto m \quad (\gamma(x) \neq \bot)} \]

\[ \text{Assgn1'} \quad \frac{\gamma \vdash \langle \sigma, e \rangle \rightarrow_e \langle \sigma, e' \rangle \quad (\gamma(x) \neq \bot)}{\gamma \vdash \langle \sigma, x := e \rangle \rightarrow_1 \sigma \quad (\gamma(x) \neq \bot)} \]

\[ \gamma \vdash \langle \sigma, e \rangle \rightarrow_e \langle \sigma, e' \rangle \]

\[ \gamma \vdash \langle \sigma, x := e \rangle \rightarrow_1 \sigma \quad (\gamma(x) \neq \bot) \]
Declarations: Little Environments

The effect of a declaration is to create a little environment which is pushed onto the existing environment. The transition relation

\[ \rightarrow_d \subseteq ((\text{Env} \times \text{Stores} \times \text{Decl}s) \times (\text{Env} \times \text{Stores})) \]

\[
\frac{\gamma \vdash \langle \sigma, \text{int } x \rangle}{\rightarrow_d \langle [x \mapsto l], [l \mapsto \bot] \sigma \rangle} \quad \text{(l \notin \text{Range}(\gamma))}
\]

\[
\frac{\gamma \vdash \langle \sigma, \text{bool } x \rangle}{\rightarrow_d \langle [x \mapsto l], [l \mapsto \bot] \sigma \rangle} \quad \text{(l \notin \text{Range}(\gamma))}
\]
Scope

- The scope of a name begins from its definition and ends where the corresponding scope ends.
- Scopes end with definitions of functions.
- Scopes end with the keyword `end` in any `let ... in ... end` or `local ... in ... end`.
- Scopes are delimited by brackets “[...]” in (fully-bracketed) λ-abstractions.
- We simply use `{}` to delimit scope.
Scope Rules

• Scopes may be disjoint
• Scopes may be nested one completely within another
• A scope cannot span two disjoint scopes
• Two scopes cannot (partly) overlap
Disjoint Scopes

let
val x = 10;
fun fun1 y = let ...
in ...
end

fun fun2 z = let ...
in ...
end

fun1 (fun2 x)
Nested Scopes

```ml
let
  val x = 10;
  fun fun1 y =
    let
      val x = 15
    in
      x + y
    end
in
  fun1 x
end
```
Overlapping Scopes

```plaintext
let
  val x = 10;
  fun fun1 y =
    ...
    ...
    ...
    ...
    fun1 (fun2 x)
end
```
Spanning

```
let
val x = 10;
fun fun1 y =
  ...
fun fun2 z =
  ...
fun1 (fun2 x)
end
```
Scope & Names

• A name may occur either as being defined or as a use of a previously defined name.

• The same name may be used to refer to different objects.

• The use of a name refers to the textually most recent definition in the innermost enclosing scope.
Names & References

let
  val x = 10; val z = 5;
  fun fun1 y =
    let
      val x = 15
    in
      x + y
    end
  end
in
  fun1 x
end

Back to scope names
Names & References

```
let
  val x = 10; val z = 5;
  fun fun1 y =
    let
      val x = 15
    in
      x + y
    end
end

fun1 x
  val x = 10;
  fun fun1  y =
    let
      val x = 15
    in
      x + y * z
    end
end
```

Back to scope names
Names & References

```ml
let
  val x = 10; val z = 5;
  fun fun1 y =
    let
      val x = 15
    in
      x + y
    end
  end
val z = 5;
* z
```

Back to scope names
Names & References

let

val x = 10; val z = 5;
fun fun1 y =

let
val x = 15
in
x + y
end

val z = 5;

* z

fun1 x

Back to scope names
let
  val \( x = 10; \) val \( z = 5; \)
  fun fun1 \( y = \)
  let
    val \( x = 15 \)
    in
      \( x + y \)
    end
  end
val \( z = 5; \)
* \( z \)

Back to scope names
Names & References

let

val \( x = 10; \) val \( z = 5; \)

fun \( \) fun1 \( y = \)

\( let \)

val \( x = 15 \)

\( in \)

\( x + y * z \)

\( end \)

\( end \)

let

\( z = 5; \)

\( * \) \( z \)

Back to scope names
Names & References

let
val \( x = 10; \) val \( z = 5; \)
fun \( \text{fun1} \)
y =
let
val \( x = 15 \)
in
end

val \( z \)

Back to scope names
Names & References

```
let
  val x = 10; val x = x - 5;
  fun fun1 y =
    let
      ...
    in
      ...
    end
  fun fun2 z =
    let
      ...
    in
      ...
    end
in
  fun1 (fun2 x)
end
```

Back to scope names
Names & References

```
let
  val x = 10; val x = x - 5;
  fun fun1 y = let ...
  in ...
  end
fun fun2 z =
  let ...
  in ...
  end
in fun1 (fun2 x)
end
```

Back to scope names
Names & References

let
  val x = 10; val x = x - 5;
  fun fun1
    y = let ...
      in ...
      end
  in
    fun1 (fun2 x)
  end

Back to scope names
Definition of Names

Definitions are of the form

\textit{qualifier name} \ldots = \textit{body}

\begin{itemize}
  \item \texttt{val name} =
  \item \texttt{fun name ( argnames )} =
  \item \texttt{local definitions in definition end}
\end{itemize}
Use of Names

Names are used in expressions. Expressions may occur

• by themselves – to be evaluated
• as the \textit{body} of a definition
• as the \textit{body} of a \texttt{let}-expression
  \begin{verbatim}
  let definitions
  in  expression
  end
  \end{verbatim}
• as the \textit{body} of a \texttt{local}-declaration
  \begin{verbatim}
  local definitions
  in  definition
  end
  \end{verbatim}
Example 17.2 local. Consider the following example ML program which uses local declarations in the development of the algorithm to determine whether a positive integer is perfect.

```ml
local
  exception invalidArg;

  fun ifdivisor3 (n, k) =
    if n <= 0 orelse
      k <= 0 orelse
      n < k
    then raise invalidArg
    else if n mod k = 0
    then k
    else 0;

  fun sum_div2 (n, l, u) =
    if n <= 0 orelse
      l <= 0 orelse
      l > n orelse
      u <= 0 orelse
      u > n
    then raise invalidArg
    else if l > u
    then 0
    else ifdivisor3 (n, l)
```

```
fun perfect n = 
  if n <= 0
  then raise invalidArg
  else
    let
      val nby2 = n div 2
    in
      n = sum_div2 (n, 1, nby2)
    end
  end
Scope & local

local

fun fun1

valid match for y = ...

fun fun2

valid match for z = ...

fun1

in

fun fun3

valid match for x = ...

fun2 ...

fun1 ...

end
Execution in the Modified Environment

Once a declaration has been processed a new scope $\gamma'$ is created in which the new variables are available for use in addition to everything else that was previously present in the environment $\gamma$ (unless it has been “hidden” by the use of the same name in the new scope). $\gamma'$ is pushed onto $\gamma$ to create a new environment $\gamma[\gamma']$. For any variable $x$,

$$\gamma[\gamma'](x) = \begin{cases} 
\gamma'(x) & \text{if } x \in \text{Dom}(\gamma') \\
\gamma(x) & \text{if } x \in \text{Dom}(\gamma) - \text{Dom}(\gamma') \\
\bot & \text{otherwise}
\end{cases}$$

\[
\begin{array}{c}
\text{D - Seq} \\
\gamma \vdash \langle \sigma, d_1 \rangle \rightarrow_d \langle \gamma_1, \sigma_1 \rangle \\
\gamma[\gamma_1] \vdash \langle \sigma_1, d_2 \rangle \rightarrow_d \langle \gamma_2, \sigma_2 \rangle \\
\gamma \vdash \langle \sigma, d_1; d_2 \rangle \rightarrow_d \langle \gamma_1[\gamma_2], \sigma_2 \rangle
\end{array}
\]
Semantics of Anonymous Blocks

\[
\begin{align*}
\text{Block} & \quad \frac{\gamma \vdash \langle \sigma, d \rangle \xrightarrow{d}^{\ast} \langle \gamma', \sigma' \rangle}{\gamma[\gamma'] \vdash \langle \sigma', c \rangle \xrightarrow{c}^{\ast} \sigma''} \\
\gamma \vdash \langle \sigma, \{d; c\} \rangle & \xrightarrow{c} \sigma'' \upharpoonright \text{Dom}(\sigma)
\end{align*}
\]

**Note.**

- Note the use of the multi-step transitions on both declarations and commands

- We have given up on single-step movements, since taking these “big”-steps in the semantics is more convenient and less cumbersome

- Note that the “little” environment \(\gamma'\) which was produced by the declaration \(d\) is no longer present on exiting the block.

- On exiting the block the domain of the state returns to \(\text{Dom}(\sigma)\), shedding the new locations that were created for the “little” environment.
Algorithm 10 A simple abstract interpreter for propositional Prolog

Require: A Prolog program $P$ and ground goal $G$
Ensure: yes if $P \vdash G$ else no

1: resolvent := \{G\}
2: while not empty(resolvent) do
3:     Choose goal $A$ from resolvent
4:     Choose a ground instance of some clause $A' \leftarrow B_1, \ldots, B_k$ from $P$ such that $A \equiv A'$
5:     if $A'$ does not exist then
6:         exit loop
7:     end if
8:     resolvent := (resolvent − \{A\}) ∪ \{B_1, \ldots, B_k\}
9: end while
10: if empty(resolvent) then
11:     return yes
12: else
13:     return no
14: end if
Algorithm 11 An abstract interpreter for general Prolog programs

Require: A Prolog program $P$ and goal $G$
Ensure: if $P \vdash G$ then $\theta$ else no

1. Standardize variables apart in $P \cup \{G\}$
2. $\text{resolventStack} := \text{empty}$
3. $\theta := 1$
4. $\text{push}(\text{resolventStack}, \theta G)$
5. while $\neg \text{empty}(\text{resolventStack})$ do
6. $A_0 := \text{pop}(\text{resolventStack})$
7. if $\exists$ clause $A' \leftarrow B_1, \ldots, B_k \in P : \text{unifiable}(A', A_0)$ then
8. $\tau := \text{mgu}(A', A_0)$
9. $\theta := \tau \circ \theta$
10. else
11. exit loop
12. end if
13. if $k > 0$ then
14. $\text{push}(\text{resolventStack}, \theta B_k, \ldots, \theta B_1)$
15. end if
16. end while
17. if $\text{empty}(\text{resolventStack})$ then
18. return $\theta$
19. else
20. return $\text{no}$
21. end if
Figure 3: Case $m > 0$ and $n > 0$