CSL 665: Introduction (Logic and Functional) Programming
I semester 2003-04

Minor II Tue 21 Oct 2003 V 344 8:00-??:?? Max Marks 38

Notational conventions.

- $\triangleq$ denotes a definition of a name by a term (usually a λ-term)
- $\equiv$ denotes "syntactic identity". It is very often used to replace a name by its definition
- $\equiv_\alpha$ denotes α equivalence.
- For any function symbol f or predicate symbol p, $\alpha(f)$ and $\alpha(p)$ denote their respective arities.

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1. Let $\Sigma = \{zero, succ, pred, plus\}$ be a signature with $\alpha(zero) = 0$, $\alpha(succ) = 1 = \alpha(pred)$ and $\alpha(plus) = 2$. Let $\mathcal{I}$ be the interpretation in $\mathbb{Z}$, the set of integers such that $\mathcal{I}(zero) = 0$, $\mathcal{I}(succ) = \lambda x[x+1]$, $\mathcal{I}(pred) = \lambda x[x-1]$, and $\mathcal{I}(plus) = \lambda(x,y)[x+y]$.

   (a) Define the set of normal forms (the subset of T_Σ).

   (b) Define rewrite rules to compute the normal form of each term.

   (c) Define the unique Σ-homomorphism $h_\mathbb{Z} : T_\Sigma \to \mathbb{Z}$.

   (d) Let $\mathcal{I}'$ be another interpretation in $\mathbb{E}$, the set of all even integers with $\mathcal{I}'(zero) = 0$, $\mathcal{I}'(succ) = \lambda x[x+2]$, $\mathcal{I}'(pred) = \lambda x[x-2]$, and $\mathcal{I}(plus) = \lambda(x,y)[x+y]$. Show that the two interpretations are isomorphic.

   $2 + 3 + 2 + 5 = 12$ marks

   <u>Solution</u>

   (a) The set of normal forms consists of *unique* representations of the integers. This yields

   $$\boxed{\{zero\} \cup \{succ^n(zero) \mid n > 0\} \cup \{pred^n(zero) \mid n > 0\}}$$

   where $succ^n$ and $pred^n$ denote their respective n-fold applications.

   (b) The equations which define the rewrite rules are as follows:

   $$\begin{array}{lcl} succ(pred(n)) & = & n \\ pred(succ(n)) & = & n \\ plus(0, n) & = & n \\ plus(succ(m), n) & = & succ(plus(m, n)) \\ plus(pred(m), n) & = & pred(add(m, n)) \end{array}$$

   However, in these rewrite rules m, n are not necessarily in normal form. Hence a function rewrite which takes this into account in a structurally inductive fashion needs to be defined. An ML implementation of this function `rewrite` could be the following:

```
fun rewrite zero = zero
  | rewrite (succ (pred (x)))    = rewrite x
  | rewrite (succ (plus (x, y))) = let val xy = rewrite (plus (x, y))
                                   in  rewrite (succ xy)
                                   end
  | rewrite (pred (succ (x)))    = rewrite x
  | rewrite (pred (plus (x, y))) = let val xy = rewrite (plus (x, y))
                                   in  rewrite (pred xy)
```

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                                    end
  | rewrite (plus (zero, y))     = rewrite y
  | rewrite (plus (succ (x), y)) = let val xn = rewrite x;
                                       val yn = rewrite y;
                                   in  rewrite (succ (plus (xn, yn)))
                                   end
  | rewrite (plus (pred (x), y)) = let val xn = rewrite x;
                                       val yn = rewrite y;
                                   in  rewrite (pred (plus (xn, yn)))
                                   end

  | rewrite (x)                  = x (* all other cases *)
```

(c) The unique Σ homomorphism $h_{\mathbb{Z}}$ is defined as follows for all $t, u \in T_\Sigma$

$$\begin{array}{lcl} h_{\mathbb{Z}}(zero) & = & 0 \\ h_{\mathbb{Z}}(succ(t)) & = & h_{\mathbb{Z}}(t) + 1 \\ h_{\mathbb{Z}}(pred(t)) & = & h_{\mathbb{Z}}(t) - 1 \\ h_{\mathbb{Z}}(plus(t,u)) & = & h_{\mathbb{Z}}(t) + h_{\mathbb{Z}}(u) \end{array}$$

(d) The interpretation $\mathcal{I}'$ defines the homomorphism $h_{\mathbb{E}}$ for all $t, u \in T_\Sigma$:

$$\begin{array}{lcl} h_{\mathbb{E}}(zero) & = & 0 \\ h_{\mathbb{E}}(succ(t)) & = & h_{\mathbb{E}}(t) + 2 \\ h_{\mathbb{E}}(pred(t)) & = & h_{\mathbb{E}}(t) - 2 \\ h_{\mathbb{E}}(plus(t,u)) & = & h_{\mathbb{E}}(t) + h_{\mathbb{E}}(u) \end{array}$$

The isomorphism between the two interpretations is defined by

$$\boxed{h_{\mathbb{E}}(t) = 2 \times h_{\mathbb{Z}}(t)}$$

and its unique inverse.

2. For any predicate p with $FV(p) = \{x_1, \ldots, x_n\}$, let the **universal closure** of p (denoted $\vec{\forall}[p]$) be the predicate $\forall x_1 \ldots \forall x_n[p]$ and similarly the **existential closure** of p (denoted $\vec{\exists}[p]$) be the predicate $\exists x_1 \ldots \exists x_n[p]$.

   A **renaming substitution** is a 1-1 and onto function $\theta : X \to X$ which is almost everywhere the identity. Let $p \equiv_\rho q$ if there exists a renaming substitution θ such that $p\theta \equiv_\alpha q\theta$.

   (a) Prove that $\equiv_\rho$ is an equivalence relation on $T_\Omega(X)$.

   (b) Show using the semantics of First-order logic that for any two predicates p and q, $p \equiv_\alpha q$ implies $\models p \leftrightarrow q$.

   (c) Does $p \equiv_\rho q$ imply $\models p \leftrightarrow q$? If so, prove it, otherwise justify your answer with an example.

   (d) Let $\mathcal{I}$ be an interpretation and p a first-order predicate. Suppose $\mathcal{I} \models p\theta$ for every renaming substitution θ. Then what can you say about the validity of the formulas $\vec{\forall}[p]$ and $\vec{\exists}[p]$ under $\mathcal{I}$? Justify your answer with a proof or a counterexample.

   $3 + 3 + 3 + 4 = 13$ marks

   <u>Solution</u>

   (a) $\equiv_\rho$ is an equivalence relation provided it is reflexive, symmetric and transitive. The proof requires that we use the corresponding properties of α-equivalence, viz. that $\equiv_\alpha$ is reflexive, symmetric and transitive. It is clear that the identity (or empty) substitution ϵ is also a renaming substitution $p\epsilon \equiv_\alpha p\epsilon \equiv_\alpha p$. Hence for any p, $p \equiv_\rho p$. The property of symmetry again follows from the symmetry of $\equiv_\alpha$ since $p \equiv_\rho q$ implies for some renaming substitution θ, $p\theta \equiv_\alpha q\theta$. Transitivity similarly follows.

(b) Let $p \equiv_\alpha q$. Then for any interpretation $\mathcal{I}$ and state s, the since the truth of p and q depends on finding the truth value of the predicate on all variants of s which involve bound variables, it is then easy to prove by structural induction that p is true in a state iff q is true. The induction cases go as follows:

$p \equiv_\alpha q \equiv A$, where A is an atomic predicate. Since there are no bound variables, $p \equiv q \equiv A$. Then clearly $\mathcal{I}, s \vDash p$ iff $\mathcal{I}, s \vDash q$, for any interpretation and state. Hence $\vDash p \leftrightarrow q$.

$p \equiv \neg p' \equiv_\alpha \neg q' \equiv q$. Clearly $p' \equiv_\alpha q'$ and by induction hypothesis we may assume $\vDash p' \leftrightarrow q'$. From the semantics of negation it follows that $\vDash p \leftrightarrow q$.

$p \equiv p_1 \circ p_2 \equiv_\alpha q_1 \circ q_2 \equiv q, \circ \in \{\wedge, \vee, \rightarrow, \leftrightarrow\}$. The induction hypothesis may be applied to yield $\vDash p_1 \leftrightarrow q_1$ and $\vDash p_2 \leftrightarrow q_2$ and then again the result follows from the semantics of $\circ$.

$p \equiv Qx[p'] \equiv_\alpha Qy[q'] \equiv q, Q \in \{\forall, \exists\}$. In this case for any interpretation $\mathcal{I}$ and state s, the truth of p requires computing over all x-variants of s and the truth value of q requires computing over all y-variants of s. In neither case does the truth value depend on $s(x)$ or $s(y)$ alone. Since $\{s' \mid s'$ is an x-variant of $s\} = \{s'' \mid s''$ is a y-variant of $s\}$ and for all other variables p and q are the same, it follows that p and q would have the same truth value for each state s in each interpretation $\mathcal{I}$. Hence $\vDash p \leftrightarrow q$.

(c) Renaming does not preserve free variables (unlike α-equivalence). Hence truth values will not be preserved under renaming. Hence the two predicates $prime(x)$ (with x as free variable) and $prime(y)$ (with y as free variable) under an interpretation such as the Positive integers in a state s such that $s(x) = 2$ and $s(y) = 4$ yield different truth values for primality.

(d) If $\mathcal{I} \vDash p\theta$ for every renaming substitution θ, then clearly the truth of the predicate is independent of state. Hence even if $FV(p) \neq \emptyset$, the free variables and their values in any state do not affect the truth of the predicate. Hence any closure (universal or existential) of the free variables has no effect on the truth of the predicate. Hence $\mathcal{I} \vDash p\theta$ implies $\mathcal{I} \vDash \vec{\forall}[p]$ and $\mathcal{I} \vDash \vec{\exists}[p]$.

3. Define interpretations on domains of discourse containing <u>at least 3 elements</u> and give a state in which the following hold. Assume P and Q are atomic predicate symbols of an arity of your choice.

(a) $\neg((P \wedge \exists x[Q]) \leftrightarrow \exists x[P \wedge Q])$

(b) $\neg((P \rightarrow \forall x[Q]) \leftrightarrow \forall x[P \rightarrow Q])$

(c) $\neg((\forall x[P] \rightarrow Q) \leftrightarrow \exists x[P \rightarrow Q])$

$4 + 4 + 5 = 13$ marks

<u>*Solution*</u> it suffices to show that

(a) $(P \wedge \exists x[Q]) \not\Leftrightarrow \exists x[P \wedge Q]$

(b) $(P \rightarrow \forall x[Q]) \not\Leftrightarrow \forall x[P \rightarrow Q]$

(c) $(\forall x[P] \rightarrow Q) \not\Leftrightarrow \exists x[P \rightarrow Q]$

We assume a 3 element set $D = \{a, b, c\}$ and two sets $\mathcal{P}, \mathcal{Q} \subseteq D$ which will be the respective interpretations of the unary predicate symbols $P(x)$ and $Q(x)$. It is sufficient to display a state σ in which one of the sides of an inequivalence is true and the other side is false.

(a) $P \wedge \exists x[Q] \not\Leftrightarrow \exists x[P \wedge Q]$.
For $\mathcal{P} = \{a\}, \mathcal{Q} = \{b\}, \sigma(x) = a$ it is easy to show from the semantics that the LHS is true in the state σ whereas the RHS is false since the two sets $\mathcal{P}$ and $\mathcal{Q}$ are disjoint.

(b) $P \rightarrow \forall x[Q] \not\Leftrightarrow \forall x[P \rightarrow Q]$.
For $\mathcal{P} = \{a, c\}, \mathcal{Q} = \{a, b\}, \sigma(x) = b$, we see that LHS is true since $P(b)$ and $\forall x[Q]$ are both false, whereas the RHS is false because $P(c) \rightarrow Q(c)$ is false.

(c) $\forall x[P] \to Q \not\Leftrightarrow \exists x[P \to Q]$.
For $\mathcal{P} = \{a, b, c\}, \mathcal{Q} = \{a\}, \sigma(x) = b$, the LHS is false because $\forall x[P]$ is clearly always true, whereas $Q(x)$ is false for σ. On the other hand, the RHS is true because there does exist a value (viz. a) such that $P(x) \to Q(x)$ is true.