CS253F: Introduction to Logic for Computer Science
I semester 2002-03

Minor 2 Mon 07 Oct 2002 WS 204 11:00-12:00 Max Marks 50

Note:

1. Mobile phones are not allowed in the examination Hall.
2. Answer in the space provided on the question paper.
3. The answer booklet you have been given is for **rough work only** and will not be collected.

Laws of logical equivalence

$(\phi \vee \psi) \vee \chi \Leftrightarrow \phi \vee (\psi \vee \chi)$	Associativity	$(\phi \wedge \psi) \wedge \chi \Leftrightarrow \phi \wedge (\psi \wedge \chi)$
$\phi \vee \psi \Leftrightarrow \psi \vee \phi$	Commutativity	$\phi \wedge \psi \Leftrightarrow \psi \wedge \phi$
$\phi \vee \phi \Leftrightarrow \phi$	Idempotence	$\phi \wedge \phi \Leftrightarrow \phi$
$\phi \vee \mathbf{ff} \Leftrightarrow \phi$	Identity	$\phi \wedge \mathbf{tt} \Leftrightarrow \phi$
$\phi \vee \mathbf{tt} \Leftrightarrow \mathbf{tt}$	Zero	$\phi \wedge \mathbf{ff} \Leftrightarrow \mathbf{ff}$
$\phi \wedge (\psi \vee \chi) \Leftrightarrow (\phi \wedge \psi) \vee (\phi \wedge \chi)$	Distributivity	$\phi \vee (\psi \wedge \chi) \Leftrightarrow (\phi \vee \psi) \wedge (\phi \vee \chi)$
$\neg(\phi \vee \psi) \Leftrightarrow \neg\phi \wedge \neg\psi$	De Morgan	$\neg(\phi \wedge \psi) \Leftrightarrow \neg\phi \vee \neg\psi$
$\phi \wedge (\neg\phi \vee \psi) \Leftrightarrow \phi \wedge \psi$	Absorption	$\phi \vee (\neg\phi \wedge \psi) \Leftrightarrow \phi \vee \psi$
$\phi \wedge (\phi \vee \psi) \Leftrightarrow \phi$	Adsorption	$\phi \vee (\phi \wedge \psi) \Leftrightarrow \phi$
$\neg\neg\phi \Leftrightarrow \phi$	Negation	$\neg\mathbf{tt} \Leftrightarrow \mathbf{ff}$, $\neg\mathbf{ff} \Leftrightarrow \mathbf{tt}$
$\phi \wedge \neg\phi \Leftrightarrow \mathbf{ff}$	Complements	$\phi \vee \neg\phi \Leftrightarrow \mathbf{tt}$
$\phi \rightarrow \psi \Leftrightarrow \neg\phi \vee \psi$	Conditional	
$\phi \leftrightarrow \psi \Leftrightarrow (\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)$	Biconditional	$\phi \leftrightarrow \psi \Leftrightarrow (\phi \wedge \psi) \vee (\neg\phi \wedge \neg\psi)$

Rules of Natural Deduction

$$(\neg - \mathsf{I})\quad \frac{\Gamma, \phi \vdash \mathbf{ff}}{\Gamma \vdash \neg\phi} \qquad (\neg - \mathsf{E})\quad \frac{\Gamma, \neg\phi \vdash \mathbf{ff}}{\Gamma \vdash \phi}$$

$$(\wedge - \mathsf{I})\quad \frac{\Gamma \vdash \phi, \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}$$

$$(\wedge - \mathsf{E1})\quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi} \qquad (\wedge - \mathsf{E2})\quad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}$$

$$(\vee - \mathsf{I1})\quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi} \qquad (\vee - \mathsf{I2})\quad \frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}$$

$$(\vee - \mathsf{E})\quad \frac{\Gamma \vdash \phi \vee \psi, \quad \Gamma \vdash \phi \rightarrow \chi, \quad \Gamma \vdash \psi \rightarrow \chi}{\Gamma \vdash \chi}$$

$$(\rightarrow - \mathsf{I})\quad \frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \qquad (\rightarrow - \mathsf{E})\quad \frac{\Gamma \vdash \phi \rightarrow \psi, \quad \Gamma \vdash \phi}{\Gamma \vdash \psi}$$

$$(\leftrightarrow - \mathsf{I})\quad \frac{\Gamma \vdash \phi \rightarrow \psi, \quad \Gamma \vdash \psi \rightarrow \phi}{\Gamma \vdash \phi \leftrightarrow \psi}$$

$$(\leftrightarrow - \mathsf{E1})\quad \frac{\Gamma \vdash \phi \leftrightarrow \psi}{\Gamma \vdash \phi \rightarrow \psi} \qquad (\leftrightarrow - \mathsf{E2})\quad \frac{\Gamma \vdash \phi \leftrightarrow \psi}{\Gamma \vdash \psi \rightarrow \phi}$$

1. Let

$$\gamma ::= A \mid (\neg\gamma) \mid (\gamma \wedge \gamma) \mid (\gamma \vee \gamma)$$

where A is any atomic propostion, be the subset of Propostional Logic that is of interest in this problem. For any formula γ in this language, let γ^d (called the **dual** of γ) be the formula obtained by replacing

- all occurrences of atoms in γ by their negations,
- all occurrences of "$\wedge$" in γ by the operator "$\vee$", and
- all occurrences of "$\vee$" in γ by the operator "$\wedge$"

(a) Prove by structural induction on γ that $\gamma \Leftrightarrow \neg\gamma^d$.

(b) Refer to the datatype defined in Problem 4. Now write an ML function `dual : Prop -> Prop` that computes the dual of a formula.

($10 + 5 = 15$ *marks)*

Solution.
γ^d may be defined by structural induction as follows:

$$\begin{array}{lcllcl} A^d & \equiv & \neg A & (\neg\delta)^d & \equiv & \neg\delta^d \\ (\delta \wedge \epsilon)^d & \equiv & \delta^d \vee \epsilon^d & (\delta \vee \epsilon)^d & \equiv & \delta^d \wedge \epsilon^d \end{array}$$

(a) The induction hypothesis we use is the following:

For any formula γ, for every subformula δ where δ is a subformula of γ, $\delta \Leftrightarrow \neg\delta^d$

Case $\gamma \equiv A$**.** Then $\gamma^d \equiv \neg A$ and $\gamma \equiv A \Leftrightarrow \neg\neg A \equiv \gamma^d$

Case $\gamma \equiv \neg\delta$**.** Then by the induction hypothesis, $\delta \Leftrightarrow \neg\delta^d$, from which we get using double negation equivalence laws $\gamma \equiv \neg delta \Leftrightarrow \neg\neg\delta^d \equiv \neg\gamma^d$.

Case $\gamma \equiv \delta \wedge / \vee \epsilon$**.** Then $\gamma^d \equiv \delta^d \vee / \wedge \epsilon^d$. By the induction hypothesis we have $\delta \Leftrightarrow \neg\delta^d$ and $\epsilon \Leftrightarrow \neg\epsilon^d$. From this and the De Morgan Laws we get $\neg\gamma^d \equiv \neg(\delta^d \vee / \wedge \epsilon^d) \Leftrightarrow \neg\delta^d \vee / \wedge \neg\epsilon^d \Leftrightarrow \delta \vee / \wedge \epsilon \equiv \gamma$

(b)

```
fun dual (ATOM a) = NOT (ATOM a)
  | dual (NOT(P)) = NOT(dual(P))
  | dual (AND (P, Q)) = OR (dual(P), dual(Q))
  | dual (OR (P, Q)) = AND (dual(P), dual(Q))
```

2. A proof system is said to be **consistent** if there exists at least one formula (in the language) which is not provable. Otherwise it is said to be **inconsistent**.

   (a) Prove that the Natural Deduction system is consistent.

   (b) Add a single axiom to the Natural Deduction system to make it *inconsistent*.

   (c) Define a different model of truth, which will ensure that the Natural Deduction system (unchanged) is sound but incomplete even with the semantics of propositional formulas unchanged.

   (d) Define a model of truth and a proof system for propositional logic which is sound and complete but inconsistent, even with the semantics unchanged.

(3 + 1 + 3 + 3 = 10 marks)

Solution.

(a) It is clear that any contradiction (or contingent) formula may be taken as an example of a formula that is not provable. For instance, any formula of the form $\phi \wedge \neg\phi$ is not provable. To show that it is not provable we simply invoke the fact that $A \wedge \neg A$ is not valid, i.e. $\not\models A \wedge \neg A$. By the soundness and completeness theorems it follows that it is not provable i.e. $\not\vdash A \wedge \neg A$.

(b) By adding a contradiction as an axiom, the Natural Deduction system allows every formula to then be proved. Assume we add the axiom $\vdash \mathbf{ff}$ as an axiom. Then any formula ϕ may then be proved by contradiction, as follows

$$\begin{array}{llll} 1. & \neg\phi & \vdash \neg\phi & A.P. \\ 2. & & \vdash \mathbf{ff} & New axiom \\ 3. & \vdash \neg\neg\phi & & 1-2 \neg - I \\ 4. & \vdash \phi & & 3, \neg\neg - E(provedinclass) \end{array}$$

(c) Consider the single-element boolean algebra with a singleton truth value 1. Then every formula is true with this model of truth. The system of Natural deduction is sound but incomplete. This is because formulas of the form $\phi \wedge \neg\phi$ are not provable, with the existing proof rules.

(d) For the model of truth considered in the previous part, it is clear that a single axiom suffices viz. $\Gamma \vdash \phi$, for all formulas ϕ. Then the proof system consisting of this single axiom is sound and complete but the system is inconsistent since every formula is provable by a one-step proof.

3. We know that it is possible to simulate every row of a truth table by means of a proof in the Natural Deduction system. Consider the formula ϕ defined by $(B \rightarrow \neg(A \rightarrow C)) \rightarrow (\neg C \rightarrow B)$. Now consider a truth assignment τ such that $\tau(A) = \tau(B) = 0$ and $\tau(C) = 1$. Give a Natural Deduction proof for the row in the truth table of ϕ that corresponds to this truth assignment.

(10 marks)

4. Let

```
datatype Prop = ATOM of string | NOT of Prop
              | AND of Prop * Prop | OR of Prop * Prop
cnf2dnf : Prop -> Prop
```

be the datatype for the language of Problem 1. Given a formula in *conjunctive normal form (CNF)*, write an efficient ML function `cnf2dnf` to transform it to *disjunctive normal form (DNF)*. The function should also be efficient in terms of the size of the *output formula*.

(15 marks)