CSL 102: Introduction to Computer Science

II semester 2004-05

Minor 1 Mon 07 Feb 2005 VI 301& VI 401 9:30-10:30 Max Marks 40

1. Answer in the space provided on the question paper.
2. The answer booklet you have been given is for **rough work only**
3. The answer booklet will not be collected.

Q1	Q2	Q3	Q4	Q5	TOTAL
7	7	7	7	12	40

Consider a *decimal* computer which has *only* the following primitive operations:

- **addition of decimal integers of arbitrary size**
- **nonnegative powers of** 10 **(i.e.** 10^m **for all integers** $m \geq 0$)
- **multiplication by nonnegative powers of 10**
- m `div` 10 **and** m `mod` 10 **for all integers** m
- **single digit multiplication** **(i.e.** $d_1 * d_2$ **for** $0 \leq d_1, d_2 \leq 9$**).**

Now answer the following questions for this particular computer keeping in mind that

1. **correctness is of paramount importance**
2. **the more time and space efficient your solution the more the marks you will be given**

1. Give a technically complete algorithmic definition of a function $mult\_by\_digit(m, d)$ which multiplies any nonnegative integer m by a single digit $d (0 \leq d \leq 9)$. *(7 marks)*

   <u>*Solution.*</u>

   We begin by giving a simple *recursive* definition and then move on to a more efficient solution.

$$rec\_mult\_by\_digit(m, d) = \begin{cases} \bot & \text{if } m < 0 \text{ or } d < 0 \text{ or } d > 9 \\ 0 & \text{if } m = 0 \text{ or } d = 0 \\ 10(rec\_mult\_by\_digit(m \texttt{ div } 10, d)) & \\ +(m \texttt{ mod } 10) * d & \text{else} \end{cases}$$

   Exercise: Prove the correctness of this definition and that its time complexity is $O(lg_{10}m)$ single-digit multiplications (with the same assumptions as in Problem 3 and its space complexity is also $O(lg_{10}m)$ (with the same assumptions as in Problem 4).

   While it is not possible to improve the time complexity, we give a different definition to yield a *constant* space requirement. The final solution therefore is

$$mult\_by\_digit(m, d) = tr\_mult\_by\_digit(m, d)$$

   where

$$tr\_mult\_by\_digit(m, d) = \begin{cases} \bot & \text{if } m < 0 \text{ or } d < 0 \text{ or } d > 9 \\ mult\_by\_digit\_iter(m, d, 0, 0) & \text{else} \end{cases}$$

   where

$$mult\_by\_digit\_iter(m, d, p, s) = \begin{cases} \bot & \text{if } m < 0 \text{ or } d < 0 \text{ or } d > 9 \\ s & \text{if } m = 0 \text{ or } d = 0 \\ mult\_by\_digit\_iter & \\ (m \texttt{ div } 10, d, p + 1, & \\ s + 10^p((m \texttt{ mod } 10) * d) & \text{else} \end{cases}$$

2. State and prove a *correctness theorem* for your solution to the above problem. *(7 marks)*

<u>*Solution.*</u>

Lemma (Correctness of *mult_by_digit_iter*). For all nonnegative integers m, p, s and digits d, $0 \leq d \leq 9$,

$$mult\_by\_digit\_iter(m, d, p, s) = s + 10^p md$$

Proof. By induction on m (PMI Version 2).

Basis. When $m = 0$, $mult\_by\_digit\_iter(0, d, p, s) = s = s + 10^p md$.

Induction Hypothesis. For $m > 0$,

$$\boxed{\text{For all } l,\ 0 \leq l < m, \text{ and all } p \geq 0,\ s \geq 0,\ mult\_by\_digit\_iter(l, d, p, s) = s + 10^p ld}$$

Induction step. For $m > 0$, $m \ \mathtt{div}\ 10 < m$ and $m = 10(m \ \mathtt{div}\ 10) + (m \ \mathtt{mod} 10)$. Hence

$$\begin{array}{lll}
& mult_by_digit_iter(m, d, p, s) & \\
= & mult_by_digit_iter(m \ \mathtt{div}\ 10, d, p+1, s + 10^p((m \ \mathtt{mod}\ 10) * d)) & \\
= & [s + 10^p((m \ \mathtt{mod}\ 10) * d)] + 10^{p+1}((m \ \mathtt{div}\ 10) * d) & \text{IH} \\
= & [s + 10^p(m \ \mathtt{mod}\ 10)d] + 10^{p+1}(m \ \mathtt{div}\ 10)d & \\
= & s + 10^p[10(m \ \mathtt{div}\ 10) + (m \ \mathtt{mod}\ 10)]d & \\
= & s + 10^p md &
\end{array}$$

QED

Theorem (Correctness of *mult_by_digit*). For all nonnegative integers m and digits d, $0 \leq d \leq 9$,

$$mult\_by\_digit(m, d) = tr\_mult\_by\_digit(m, d) = md$$

Proof. If $m = 0$, it is clear that for all digits d, $0 \leq d \leq 9$,

$$mult\_by\_digit(0, d) = tr\_mult\_by\_digit(0, d) = 0 * d = 0$$

For $m > 0$, $0 < d \leq 9$ we have

$$\begin{array}{lll}
& mult_by_digit(m, d) & \\
= & tr_mult_by_digit(m, d) & \\
= & mult_by_digit_iter(m, d, 0, 0) & \\
= & 10^0 md + 0 & \text{Correctness of } mult_by_digit_iter \\
= & md &
\end{array}$$

QED

3. Assume that single-digit-multiplication is the most expensive operation on this computer and that all other operations are insignificant and take no time at all. State and prove a theorem giving the time-complexity of your algorithm in units of single-digit-multiplications.

Theorem (Correctness).

Proof. *(7 marks)*

<u>*Solution.*</u>

Lemma. $\mathcal{T}(mult\_by\_digit\_iter(m, d, p, s)) \leq k+1$, where $10^k \leq m < 10^{k+1}$, i.e. $k+1$ is the number of digits in the decimal representation of m.

Proof. By induction on $k \geq 0$.

Basis. For $k = 0$ it is clear that m is a single digit number. Hence m `div` $10 = 0$ and m `mod` $10 = m$. Hence

$$\begin{array}{ll} & mult\_by\_digit\_iter(m, d, p, s) \\ = & mult\_by\_digit\_iter(0, d, p+1, s + 10^p(m * d)) \\ = & s + 10^p(m * d) \end{array}$$

which involves just one single digit multiplication. Hence $\mathcal{T}(mult\_by\_digit\_iter(m, d, p, s)) = 1 \leq k + 1$.

Induction hypothesis. For some $k > 0$ and all d, p, s and m, $10^k \leq m < 10^{k+1}$,

$$\boxed{\mathcal{T}(mult\_by\_digit\_iter(m, d, p, s)) \leq k + 1}$$

Induction step. Let $10^{k+1} \leq m < 10^{k+2}$. Then $10^k \leq m \texttt{ div } 10 < 10^{k+1}$ and $0 \leq m \texttt{ mod } 10 \leq 9$. Then we have

$$\begin{array}{lll} & \mathcal{T}(mult\_by\_digit\_iter(m, d, p, s)) & \\ = & \mathcal{T}(mult\_by\_digit\_iter(m \texttt{ div } 10, d, p+1, s + 10^p((m \texttt{ mod } 10) * d))) & \\ \leq & (k+1) + 1 & \text{IH and } (*) \\ = & k + 2 & \end{array}$$

QED

Theorem. $\mathcal{T}(mult\_by\_digit(m, d)) = O(\lceil lg_{10} m \rceil)$ *Proof.* If $10^k \leq m \texttt{ div } 10 < 10^{k+1}$ then $\lceil lg_{10} m \rceil = k + 1$

$$\begin{array}{lll} & \mathcal{T}(mult\_by\_digit(m, d)) & \\ = & \mathcal{T}(mult\_by\_digit\_iter(m, d, 0, 0)) & \\ \leq & \lceil lg_{10} m \rceil & \text{By the previous Lemma} \\ = & O(\lceil lg_{10} m \rceil) & \end{array}$$

QED

4. Assume that each integer occupies a single cell of memory. State and prove a theorem giving the space complexity of your algorithm in units of memory cells.

(7 marks)

Solution.

Lemma. $\mathcal{S}(mult\_by\_digit\_iter(m, d, p, s)) \leq k$, for some constant $k > 0$.

Proof. By induction on $m \geq 0$.

Basis. For $m = 0$ it is clear that $\mathcal{S}(mult\_by\_digit\_iter(m, d, p, s)) \leq k$

Induction hypothesis. For all $n < m$ and all d, p, s and some constant k,

$$\boxed{\mathcal{S}(mult\_by\_digit\_iter(n, d, p, s)) \leq k}$$

Induction step. We have

$$\begin{array}{ll} & \mathcal{S}(mult\_by\_digit\_iter(m, d, p, s)) \\ = & \mathcal{S}(mult\_by\_digit\_iter(m \texttt{ div } 10, d, p+1, s + 10^p((m \texttt{ mod } 10) * d))) \\ \leq & k \end{array}$$

QED

Theorem. $\mathcal{S}(mult\_by\_digit(m, d)) = O(1)$ *Proof.* When $m > 0$ we have

$$\begin{array}{lll} & \mathcal{S}(mult\_by\_digit(m, d)) & \\ = & \mathcal{S}(mult\_by\_digit\_iter(m, d, 0, 0)) & \\ \leq & k & \text{for some constant } k \text{, by the previous Lemma} \\ = & O(1) & \end{array}$$

QED

5.

6. (a) Use the function *mult_by_digit* to give a technically complete algorithmic definition of the function $prod(m, n)$ which computes the product of any two integers m, n.

$$prod(m,n) = \begin{cases} 0 & \text{if } m = 0 \text{ or } n = 0 \\ prod_iter(m,n,0,0) & \text{else} \end{cases}$$

where

$$prod_iter(m,n,p,s) = \begin{cases} s & \text{if } m = 0 \text{ or } n = 0 \\ prod_iter(m, n \texttt{ div } 10, p+1, & \\ s + 10^p(mult\_by\_digit(m, n \texttt{ mod } 10))) & \text{else} \end{cases}$$

(b) What is the time-complexity of your algorithm in units of *mult_by_digit* operations? **No need to prove!**

$\mathcal{T}(prod(m,n)) = O(lg_{10}m)$

(c) What is the time complexity of your algorithm in units of "single-digit-multiplications"? **No need to prove!**

$\mathcal{T}(prod(m,n)) = O((lg_{10}m)((lg_{10}n))$

(d) What is the space complexity of your algorithm? **No need to prove!**

$\mathcal{S}(prod(m,n)) = O(1)$