CSL105: Discrete Mathematical Structures
I semester 2007-08

Tutorial sheet: **Basics of Graph Theory**
Last updated Mon Oct 1 09:58:56 IST 2007

---

1. The *towers of hanoi* is a problem whose solution may be modelled as a problem of state transitions. Consider the towers of Hanoi problem with 3 rings and model all the transition possibilites as a directed graph.

2. Many recreational problems may be regarded as graph problems for the purpose of modelling. *Given 3 uncalibrated vessels A, B and C of capacities 8 litres, 5 litres and 3 litres respectively. Vessel A is filled to the brim with a liquid. It is required to divide the liquid in the vessel A into 2 equal volumes using the other two vessels, without adding or dropping any more liquid*

   (a) Model the problem as a directed graph on *states* with transitions being the act of transferring an amount of liquid from the one vessel to another so that the source vessel becomes empty or the target vessel becomes full.

   (b) Show that there is a directed path from the *initial state* viz. the one in which vessel A is full and the others empty to a state in which the liquid is divided into two vessels which contain 4 litres each.

   (c) What are the invariant properties satisfied by the states that are reachable from the initial state? Are there any states satisfying all these invariant properties, which is not reachable?

3. Prove that in a directed or undirected graph of n vertices, if there is a path from vertex u to v then there is a path of length no more than $n-1$ edges from u to v.

4. Prove that a simple graph with n vertices must be connected if it has more than $(n-1)(n-2)/2$ edges.

5. Prove that a connected undirected graph remains connected after removing a particular edge e, if and only if e is in some circuit of the graph.

6. The **union** of two undirected graphs $G_1 = \langle V_1, E_1\rangle$ and $G_2 = \langle V_2, E_2\rangle$, denoted $G_1 \cup G_2$ is a third graph $G_3 = \langle V_3, E_3\rangle$ such that $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$. Similarly the **intersection** $G_1 \cap G_2$ is the graph consisting of only those vertices and edges that are in both G_1 and G_2. The **xor** graph of G_1 and G_2 (denoted $G_1 \oplus G_2$) is the graph $G_3 = \langle V_3, E_3\rangle$ such that $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \oplus E_2$.

   (a) Prove that each of the three operations on undirected graphs is *commutative*.

   (b) Prove that if G_1 and G_2 are *edge-dsijoint* then $G_1 \oplus G_2 = G_1 \cup G_2$.

   (c) Prove that union and intersection are *idempotent* operations and $G \oplus G$ is a *null graph*.

7. A graph G is said to be **decomposed** into two subgraphs G_1 and G_2 if $G_1 \cup G_2 = G$ and $G_1 \cap G_2$ is a null graph. Prove that a connected undirected graph is an Euler graph *iff* it can be decomposed into circuits.

8. The notions of Eulerian paths and circuits may be extended to *directed* graphs as well. An **Euler digraph** is a directed graph with no isolated vertices, which has a directed Eulerian circuit.

   (a) Prove that a directed graph (with no isolated vertices) is an Euler digraph *iff* it is strongly or weakly connected and $\delta^+(v) = \delta^-(v)$ for every vertex v

   (b) Prove that a directed graph (with no isolated vertices) has a directed Eulerian path or circuit *iff* it is strongly or weakly connected and

   - $\delta^+(v) = \delta^-(v)$ for every vertex v or
   - $\delta^+(v) = \delta^-(v)$ for every vertex v except two vertices u and w for which $\delta^+(u) = \delta^-(u) + 1$ and $\delta^-(w) = \delta^+(w) + 1$.