CSL105: Discrete Mathematical Structures
I semester 2008-09
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Tutorial sheet: **Algebra of Relations and Functions**

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1. Prove that
   (a) for any equivalence relations $\sim_1$ and $\sim_2$ on a set $A \neq \emptyset$, $\pi_{\sim_1} \sqsubseteq \pi_{\sim_2}$ <u>if and only if</u> $\sim_1 \subseteq \sim_2$.
   (b) the set Π of all partitions of a set is partially ordered by the refinement relation.

2. Prove that relational composition is monotonic, i.e. for any binary relations $R, R' \subseteq A \times B$ and $S, S' \subseteq B \times C$, if $R \subseteq R'$ and $S \subseteq S'$ then $R; S \subseteq R'; S'$.

3. Prove that for any binary relations R and S on a set A,
   (a) $(R^{-1})^{-1} = R$
   (b) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
   (c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
   (d) $(R - S)^{-1} = R^{-1} - S^{-1}$

4. Let A, B, C, D be sets and let $R, R_1, R_2 \subseteq A \times B$, $S, \subseteq B \times C$, and $T \subseteq C \times D$ be binary relations. Let ; denote the relational composition operation, where the relation $R; S \subseteq A \times C$ is defined as $R; S = \{(a, c) \mid \exists b \in B : (a, b) \in R \text{ and } (b, c) \in S\}$ Prove or disprove[1] the following properties of relational composition. Where set equality or equivalence does not hold explore whether either one side of the equality is a subset or implies the other.

   Left Monotonicity. $R_1 \subseteq R_2 \Rightarrow R_1; S \subseteq R_2; S$

   Right Monotonicity. $S_1 \subseteq S_2 \Rightarrow R; S_1 \subseteq R; S_2$

   Associativity. $(R; S); T = R; (S; T)$

   Left Distributivity over union. $R; (S_1 \cup S_2) = (R; S_1) \cup (R; S_2)$.

   Right Distributivity over union. $(R_1 \cup R_2); S = (R_1; S) \cup (R_2; S)$.

   Left Distributivity over intersection. $R; (S_1 \cap S_2) = (R; S_1) \cap (R; S_2)$.

   Right Distributivity over intersection. $(R_1 \cap R_2); S = (R_1; S) \cap (R_2; S)$.

5. Prove that for any binary relations R and S on a set A,
   (a) $(R^{-1})^{-1} = R$
   (b) $(R \cap S)^{-1} = R^{-1} \cap S^{-1}$
   (c) $(R \cup S)^{-1} = R^{-1} \cup S^{-1}$
   (d) $(R - S)^{-1} = R^{-1} - S^{-1}$

6. Prove or disprove[2] that relational composition satisfies the following distributive laws, where $R \subseteq A \times B$ and $S, T \subseteq B \times C$.
   (a) $R; (S \cup T) = (R; S) \cup (R; T)$
   (b) $R; (S \cap T) = (R; S) \cap (R; T)$
   (c) $R; (S - T) = (R; S) - (R; T)$
   (d) $(R; S)^{-1} = (S^{-1}); (R^{-1})$

7. Consider the set $\mathbf{2}^{A \times A}$ of all binary relations on a set $A \neq \emptyset$. Prove that

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[1] A property is disproved only by finding an example of an appropriate relation(s) which satisfy all the assumptions but do not satisfy the property

[2] that is, find an example of appropriate relations which actually violate the equality

(a) $\mathbf{2}^{A\times A}$ is a monoid under relational composition. What is the identity element of the *monoid*?

(b) Given that $\mathbf{2}^{A\times A}$ is closed under the relational inverse operation, show why it is not a *group*?

8. Let R be a binary relation on $A \neq \emptyset$. Consider the relations $R^+ = \bigcup_{n\in\mathbb{P}} R^n$ and $R^* = \bigcup_{n\in\mathbb{N}} R^n$, where $R^0 = Id_A$ and $R^{m+1} = R; R^m$, for each $m \in \mathbb{N}$.

   (a) Prove that R^* is a preorder[3].

   (b) Under what conditions, is it guaranteed that

      i. R^+ is irreflexive?

      ii. R^+ is asymmetric?

      iii. R^* is an equivalence?

      Prove your answer in each case.

   (c) **Fixpoint** Prove that R^* is a solution of the equation $\boxed{Id_A \cup X; R = X}$.

      Least fixpoint. Prove that R^* is the smallest set (under the $\subseteq$ ordering) solution.

      Closure Prove that $(R^*)^* = R^*$.

      Other fixpoints Let $S \subseteq A\times A$ be any other solution of the equation. Under what conditions would S be different from R. Give an example of a relation R and a relation S such that S is a solution of the equation but $S \neq R^*$.

9. Let $f : A \to B$ be a total function where A and B are nonempty sets. For any set X, let Id_X denote the identity function on X. Prove that

   (a) $f(X \cup Y) = f(X) \cup f(Y)$ for all $X, Y \subseteq A$

   (b) $f(X \cap Y) \subseteq f(X) \cap f(Y)$ for all $X, Y \subseteq A$

   (c) If f is injective then $f(X \cap Y) = f(X) \cap f(Y)$ for all $X, Y \subseteq A$

   (d) $f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y)$ for all $X, Y \subseteq A$

   (e) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$ for all $X, Y \subseteq A$

   (f) f is injective *iff* for all $X \subseteq A$, $f^{-1}(f(X)) = X$

   (g) f is surjective *iff* for all $Y \subseteq B$, $f(f^{-1}(Y)) = Y$

   (h) f is injective *iff* there exists $g : B \to A$ such that $g \circ f = Id_A$ (g is called the *left inverse* of f),

   (i) f is surjective *iff* there exists $g : B \to A$ such that $f \circ g = Id_B$ (g is called the *right inverse* of f),

   (j) f is bijective *iff* there exists $g : B \to A$ such that $f \circ g = Id_B$ and $g \circ f = Id_A$ (g is called the *inverse* of f and denoted f^{-1}).

10. Let $f : A \to B$ and $g : B \to C$ be total functions where A, B and C are nonempty sets. Prove that

   (a) f and g are both injective then so is $g \circ f$,

   (b) If f and g are both surjective then so is $g \circ f$,

   (c) If f and g are both bijective then so is $g \circ f$,

   (d) If $g \circ f$ is injective then f is injective,

   (e) If $g \circ f$ is surjective then g is surjective,

   (f) If $g \circ f$ is injective and f is surjective then g is injective,

   (g) If $g \circ f$ is surjective and g is injective then f is surjective,

[3] R^* is called the *reflexive transitive closure* of R