Brief Announcement: On Cost Sharing Mechanisms in the Network Design Game

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ABSTRACT

A fundamental network design problem is the one of Steiner Network Design. The goal is to design a network which is able to support a unit flow for each commodity, at a time, between its source-sink pair. When the flows are unsplittable, this corresponds to the Steiner forest problem and to the problem of sharing cost of the multicast by different users. As a result of greedy selfish behavior of users in the network design game, the overall quality of the resulting solution is often not as good as the globally optimum solution of the underlying problem. We are therefore interested in the problem of designing distributed cost sharing mechanisms that induce the selfish agents to converge to the near-optimum solutions.

Our main contribution is showing that $1+\epsilon$ ratio can be achieved by (non-obvious) unfair cost sharing mechanism, at least for the fractional version of the problem. Our second contribution is showing how to implement our cost sharing mechanism which guarantees fast convergence to a near-optimum equilibrium. We show that for the fractional network design problems, there are indeed such mechanisms that induce greedy agents to converge to $(1+\epsilon)$-approximate equilibria in linear time.

Categories and Subject Descriptors

F.2.2 [**Analysis of Algorithms and Problem Complexity**]: [Nonnumerical Algorithms and Problems]

General Terms

algorithms, theory

Keywords

network design, cost sharing

1. INTRODUCTION

We consider the problem of network design where the agents involved interact in a distributed, uncoordinated, and selfish manner that has received fair amount of attentions in computer science community recently, see e.g., [1, 8, 2]. As a result of greedy behavior of users, the overall quality of the resulting solution is often not as good as the globally optimum solution of the underlying problem. We are therefore interested in the problem of designing distributed cost sharing mechanisms that induce the selfish agents to converge to the near-optimum solutions.

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A fundamental network design problem is the following Steiner Network Design problem. Consider k commodities, each specified by a pair of source-sink nodes. The goal is to design a network which is able to support a unit flow for each commodity, at a time, between its source-sink pair. When the flows are unsplittable, this corresponds to the Steiner forest problem. This problem is closely related to the problem of sharing cost of multicast by different users.

The question of bounding the quality of outcomes resulting from selfish behaviors in the network design games have been studied in the past (see, e.g., Anshelevich et al. [1]). They have considered computing price of anarchy and price of stability in a so-called *fair cost allocation* mechanism. The fair cost allocation is the one in which the cost of any edge is shared *equally* among all the commodities that use that edge. (It is also called "Shapley allocation".) With fair cost allocation, Anshelevich et al. [1] show that for the Steiner tree problem, there *exist* equilibria that are $\Omega(\log k)$ factor worse that the optimum solution, where k is the total number of commodities. They also show an instance where the bound of $\log k$ is tight. Their result, furthermore, is existential: there is no evidence of a concrete mechanism that will actually induce the agents to achieve this bound. In fact, if the agents just take turns making greedy moves, then there are network examples on which the fair cost sharing proposed in [1] gives $\Omega(k)$ approximation ratio, which is the worst possible ratio that can be achieved by greedy users (each user optimizes for itself; there is no cooperation or sharing). So the natural question is whether there exists an *unfair* cost sharing mechanism that will guarantee better ratio than $\Omega(k)$. It is worth pointing out that [1] considers the harder, integer version of the network design problem. However, this is the best known result for easier fractional version of the problem as well.

More importantly, as pointed out by Fisher et al. [4, 5], there is an inherent problem with applying existing game theory work, such as work on selfish network design games [1, 8, 2] to concurrent distributed system of selfish agents. The interpretation of the Nash equilibrium framework is envisioning an idealized network model where all re-routings occur sequentially, e.g., coordinated by a central server (this is unimaginable for large networks such as the Internet). Given an arbitrary mechanism that supposedly induces a good equilibrium, it is not obvious that a concurrent distributed system of selfish agents is *ever* going to converge to any equilibrium without coordination by central server. Thus, dynamics of distributed systems does not correspond to sequential behavior of congestion games.

A sequential algorithm for fractional Steiner forest problem was designed by Garg and Khandekar [6] where flows of all commodities increase from zero simultaneously and there is no re-routing involved. The "experts" theorem of [7] is used for each edge in showing that the cost of an edge is almost completely recoverable from the "payments" done to that edge by all commodities. Our cost-subdivision is motivated by work in [6]. Our proof technique is self-contained and is completely different from [6] (not based on the above experts theorem). Our mechanism starts from arbitrary flows and lets the selfish agents continuously re-route their flows.

Our main contribution is in showing that $(1+\epsilon)$ ratio can be achieved by (non-obvious) unfair cost sharing mechanism, at least for the fractional version of the problem. Our second contribution is in considering the concurrency aspect of network design game. We show how to implement our cost sharing mechanism which guarantees polynomial-time convergence to a near-optimum equilibrium for the fractional network design problems. This requires introducing the notion of "speed limits", similar to the flow control mechanisms such as TCP. The essence of speed limit is enforcing concurrency control and preventing oscillations that could occur when many flows concurrently rush to the shortest path. Since in the network design game the flows do not "add up", and the only thing that matters is the maximal flow sent by a user, rather than the sum of flows from all users, these concurrency issues are of very different character than those considered in the concurrent congestion games [5, 3].

2. CONCURRENT NETWORK DESIGN

We consider the following network design game. Let $G=(V,E)$ be a (directed or undirected) network with non-negative edge costs w_e. Let $m=|E|$. Suppose that there are k agents $i\in[k]$, each interested in routing unit flow between its source $s_i\in V$ and sink $t_i\in V$. The flow of commodity i can be split along several paths p between s_i and t_i. Let $\mathcal{P}_i$ denote the set of paths between s_i and t_i. Let f_p denote the flow of commodity i on path $p\in\mathcal{P}_i$. Let $f_e^i=\sum_{p\in\mathcal{P}_i:e\in p} f_p$ denote the flow of commodity i along edge e. To support f_e^i units of flow of each commodity i along e, we assume that the edge e needs to be "opened" to an extent of $x_e=\max_i f_e^i$ and it costs $w_e x_e$ to do so. The social cost of the solution is $\sum_e w_e x_e$. The optimum solution that minimizes this social cost can be formulated as the following linear program.

$$\begin{array}{llcll} \min & \sum_{e\in E} w_e x_e & & & \\ \text{s.t.} & \sum_{p\in\mathcal{P}_i} f_p & \geq & 1 & \forall i\in[k] \\ & \sum_{p:e\in p\in\mathcal{P}_i} f_p & \leq & x_e & \forall i\in[k], e\in E \\ & f_p, x_e & \geq & 0 & \forall p, e \end{array} \qquad (1)$$

A *cost-sharing* mechanism is a way for each edge e to sub-divide its cost w_e among the commodities based on the current flows f_e^i of different commodities through it. We assume that each agent associated with commodities is selfish, in that, it routes (augments or re-routes) its flows along minimum-cost paths under the cost subdivision corresponding to that commodity. The objective of this paper is to design a mechanism such that the selfish agents are made to converge to a near-optimum network design solution. In order to restrict the adverse effects of concurrency, we impose some "speed-limit" constraints on the selfish agents that limit how much flow an agent can re-route in a single step. We show (Section 2.1) that there exists a cost-sharing mechanism for the fractional network design problem such that, starting from an arbitrary solution, the selfish agents (subject to some natural constraints) converge to $(1+\epsilon)$-approximate solution in rounds that are polynomial in the size of the network and polylogarithmic in the number of commodities.

2.1 Our cost sharing mechanism

For a commodity $i\in[k]$, let $\mathbf{f}^i=\{f_e^i\}_{e\in E}$ denote a unit flow from s_i to t_i. To support such flow for each commodity, the edge e has to be opened to an extent $x_e=\max_i f_e^i$. Our cost sharing mechanism sub-divides the per-unit-flow cost w_e of e into per-unit-flow costs w_e^i for commodities $i\in[k]$. This sub-division satisfies $w_e=\sum_{i\in[k]} w_e^i$ and is a function of the flow $\mathbf{f}=\{\mathbf{f}^i\}_{i\in[k]}$. The subdivision is given by

$$w_e^i(\mathbf{f}) = w_e\cdot\frac{\phi(f_e^i)}{\sum_{j\in[k]}\phi(f_e^j)} \qquad (2)$$

where ϕ is a "sufficiently" fast growing function. For example, we can use an exponential function $\phi(x)=k^{\frac{\gamma\cdot x}{\epsilon}}$ where $\gamma=O(m\log^{O(1)}k/\epsilon^{O(1)})$. This choice is same as the one in [6]. Let $\beta=\frac{\epsilon^2}{\gamma\cdot\log k}$ and note that increasing or decreasing flow f_e^i by at most β for each i changes the the cost w_e^i by at most $(1+\epsilon)$ factor.

Speed limit. We impose the constraint that any agent i can change the flow f_e^i by an amount at most β in a single step.

Note that for any fast-growing function ϕ, if the flows of different commodities on an edge e are roughly equal, the cost w_e is roughly equally sub-divided into those commodities. On the other hand, if the flow of a commodity i exceeds the flow of other commodities by a large amount, this commodity bears almost the entire cost $w_e^i\approx w_e$.

The costs in equation (2) denote the per-unit-flow costs. Thus a flow $\mathbf{f}^i$ of agent i "pays" a cost of $w_e^i f_e^i$ to edge e and a cost of $\sum_e w_e^i f_e^i$ overall. The agents therefore re-route their flows to minimize their costs subject to the speed limit constraints.

THEOREM 2.1. *There exist cost share mechanisms for which starting from an arbitrary solution, the selfish agents subject to some speed-limit constraints that limit how much flow can be re-routed in a single step, converge to $(1+\epsilon)$-approximation in*

$$\tilde{O}\left(m\log^{O(1)}k/\epsilon^{O(1)}\right)$$

iterations.

The proof is based on showing that some notion of "aggregate" equilibrium, which implies near-optimality, is reached in the given number of rounds.

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