# Coalitional Games on Graphs: Core Structure, Substitutes and Frugality

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## Categories and Subject Descriptors

K.4.4 [**Computing Milieux**]: Computers and Society—*Electronic Commerce*; J.4 [**Computer Applications**]: Social and behavioral sciences

## General Terms

Algorithms, Design, Economics.

## Keywords

Game theory, Core, Frugality, VCG Mechanism, Lattice, Agents, Substitutes.

## 1. PRELIMINARIES

We study mechanisms that can be modeled as coalitional games with transferable utilities, and apply ideas from mechanism design and game theory to problems arising in a network design setting. We establish an equivalence between the game-theoretic notion of agents being *substitutes* [3] and the notion of *frugality* [1] of a mechanism. We characterize the core of the network design game and relate it to outcomes in a sealed bid auction with VCG payments — we show that in a game, agents are substitutes if and only if the core [7] of the game forms a complete lattice.

Proofs are omitted due to the lack of space. Interested readers are referred to [5].

While we consider minimization games on graphs, our results extend to maximization games in a straightforward manner. We consider a graph $G = (V, E)$, each of whose edges is controlled by an autonomous agent. The *auctioneer* (also termed the *buyer*) is endowed with a sum $U$ of money. The auctioneer wishes to purchase a collection $S$ of edges from the respective agents such that $S$ induces a subgraph with some desirable property $P$ (we refer to $S$ as a *solution*). For instance, in the Minimum Spanning Tree (or MST) game, the auctioneer wishes to acquire a collection of edges that constitute a spanning tree of $G$. Each edge $e_i$, and thus the corresponding agent $i$, has an associated cost $C_i$, such that the agent incurs cost $C_i$ if $e_i$ is (bought and) used by the auctioneer. The value of $C_i$ is known only to $i$. Each agent receives a payment $P_i$ from the auctioneer, such that $\sum_i P_i \leq U$. Clearly $P_i$ must be no less than $C_i$ if $e_i \in S$; WLOG, we assume that $P_i = 0$ otherwise. All parties aim to maximize their *payoff* – the payoff of the auctioneer is $U - \sum_i P_i$, while that of any other agent $i$ is $P_i - C_i$. Thus, the auctioneer's objective is to construct a solution at the lowest possible cost. We assume the sum $U$ is no less than the cost of the second best solution.

First, we set out some definitions and formalism. After [7], we define

Definition 1. *A* Coalitional game $< N, V >$ with transferable payoff *consists of a finite set $N$, the* set of players*, and a function $V$ that associates with every nonempty subset (*coalition*) $S$ of $N$, a real number $V(S)$ (the* worth *of $S$).*

For each coalition $S$, $V(S)$ is the total payoff that is available for division among the members of $S$. For any vector $(x_i)_{i \in N}$ of real numbers (*profile*), let $x(S) = \sum_{i \in S} x_i$. $(x_i)_{i \in S}$ is an *$S$-feasible payoff vector* if $x(S) = V(S)$. An $N$-feasible payoff vector is a *feasible payoff profile*.

In a coalition game with transferable utilities, the *core* of the game denotes the set of stable outcomes of the game.

Definition 2. *The* core *of coalitional game $< N, v >$ is the set of feasible payoff profiles $(x_i)_{i \in N}$ for which $V(S) \leq x(S)$ for every coalition $S$.*

As noted above, in our model, there is one auctioneer and multiple agents; and each agent owns a resource and the auctioneer needs a collection of resources for forming a solution $S$. The set $N$ of all players consists of the auctioneer and the agents. We denote the auctioneer as agent 0. We note that the *worth* of a subset of $S$ may be zero (if that subset does not by itself constitute a solution). In other words, the function $V$ may exhibit *complementarity*. We assume that for any $S_1 \subseteq S_2$, $V(S_2) \geq V(S_1)$; that is, $V$ possesses the "*zero cost of disposal*" property. We specify $V(S)$ for all $S \subseteq N$ in the following manner. $V(S) = 0$ if

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*EC'03,* June 9–12, 2003, San Diego, California, USA.

$0 \notin S$, or if $(S - \{0\})$ does not contain a feasible solution. Otherwise, $V(S) = U - C(S)$, where $C(S)$ is the cost of the best solution contained in $S$. Let the cost of an optimal solution be $\mathcal{C}$; that is, $V(N) = U - \mathcal{C}$.

Let $\{O_1, \cdots, O_m\}$ be the set of all[1] optimal solutions. Define $\mathcal{O} = \cap_{l=1}^{m} O_l$, the set of agents that are contained in every optimal solution.

The "*agents are substitutes*" property (or the *substitutes* property, for short) is a commonly employed characterization. After [3], we define:

DEFINITION 3. *We say that* agents are substitutes *if* $V(N) - V(N-K) \geq \sum_{i \in K}(V(N) - V(N - \{i\}))$ *for all* $K \subset N$ *such that* $0 \notin K$.

# 2. COST OF VCG PAYMENT

The VCG mechanism [4, 6, 9] is a celebrated incentive-compatible (truth-revealing) strategy which is widely used. The costliness of the VCG solution, though, varies with the problem in hand. Characterizing the *frugality* of the VCG payment is important to determine whether a VCG based strategy is practical or not.

The concept of frugality is introduced (albeit without a formal definition) in [1]. We formalize the notion of frugality and relate it to well-studied concepts in game theory. A different treatment of frugality is offered in [8].

DEFINITION 4. *We define the* frugality ratio *of a payoff vector* $\pi$ *as* $\mathcal{F}(\pi) = \max_{O \subseteq \mathcal{O}} \frac{\sum_{i \in O} \pi_i}{V(N) - V(N-O)}$. *We say that* $\pi$ *is* frugal *if* $\mathcal{F}(\pi) \leq 1$.

Consider $\pi^V$, the VCG payoff vector. It is easy to show that

$$\pi_i^V = \begin{cases} V(N) - V(N - \{i\}) & \text{if } i \in \mathcal{O}, \\ 0 & \text{otherwise.} \end{cases}$$

Our first result is the following theorem:

THEOREM 1. $\pi^V$ *is* frugal *if and only if agents are substitutes.*

# 3. STRUCTURE OF THE CORE

Core structure has been extensively studied in literature. Bikhchandani and Ostroy [2] show that if buyers are substitutes, then the core is a lattice with respect to the buyers. They also show the equivalence of "buyers are substitutes", $\pi^V$ being in the core, and $\pi^V$ being the maximum of all payoffs in the core.

Let $CORE$ denote the core of the game, and $\pi \in CORE$. Note that if $j \notin \mathcal{O}$ then $\pi_j = 0$, and that $\pi_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \pi_j$ is the payoff of agent 0.

Let $\pi^1, \pi^2 \in CORE$, and let $\overline{\pi}$ be defined as follows – $\forall i \in (N - \{0\})$, $\overline{\pi}_i = \max(\pi_i^1, \pi_i^2)$; and $\overline{\pi}_0 = U - \mathcal{C} - \sum_{j \in N - \{0\}} \overline{\pi}_j$. It can be shown that $\overline{\pi}_0$ is always non-negative.

The following result is due to Bikhchandani and Ostroy [2]:

LEMMA 1. *If* $V(N) - V(N - \mathcal{O}) \geq \sum_{i \in \mathcal{O}}(V(N) - V(N - \{i\}))$ *and* $\pi^1, \pi^2 \in CORE$*, then* $\overline{\pi}, \underline{\pi} \in CORE$.

Our main result regarding core structure is the following lemma:

[1]Note that $\forall i \in O_l$, $O_l - \{i\}$ is not a solution.

LEMMA 2. *If* $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \underline{\pi} \in CORE$*, then agents are substitutes.*

We say $\pi^1 \preceq \pi^2$ if $\forall i \in N - \{0\}, \pi_i^1 \leq \pi_i^2$.

LEMMA 3. $(CORE, \preceq)$ *is a lattice if and only if* $\pi^1, \pi^2 \in CORE \Rightarrow \overline{\pi}, \underline{\pi} \in CORE$.

Lemmas 1, 2 and 3 imply the following:

THEOREM 2. *A core is a lattice if and only if the agents are substitutes.*

# 4. CONCLUSIONS

This note correlates some notions in mechanism design to concepts from game theory. We formalize the notion of frugality [1] and extend it to define the frugality ratio of a mechanism. We establish the equivalence of frugality and a slightly modified version of the *agents are substitutes* [3] condition. We show that if the core forms a lattice, then the agents satisfy the substitutes condition. This leads to a three-way equivalence of the core being a lattice, agents satisfying the substitutes condition, and the VCG payment being frugal.

Archer and Tardos [1] showed that the VCG payment in the Shortest Path game can be very high as compared to the cost of second best solution. On the other hand, the VCG payment in games like MST coincide with the cost of the second best solution [5]. This can be attributed to the fundamental difference in the structure of the core. The core of the MST game is a lattice, implying that within the core, the agents are in some sense independent. On the other hand, the core for the shortest path game does not satisfy the lattice property, and a change in the bid of one agent affects other agents even inside the core. Thus, the basic structure of the core can provide an insight into why the particular mechanism (in this case, *VCG*) exhibits very different characteristics, in terms of frugality of payments, from one game to another.

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