Market Equilibrium using Auctions for a Class of Gross-Substitute Utilities

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Abstract. In addition to useful Economic interpretation, auction based algorithms are generally found to be efficient. In this note, we observe that the auction-based mechanism can also be used to efficiently compute market equilibrium for a large class of utility functions satisfying gross substitutability, including a range of CES (constant elasticity of substitution) and Cobb-Douglas functions.

1 Introduction

The Market Equilibrium model is a classical problem in micro-economics. In the late nineteenth century two market models have been studied, termed the Fisher model [2] and the more general Walrasian model [14]. Given a set of goods and a set of buyers who have utility for the goods, the problem is to determine prices and allocation of the goods such that no buyer is induced to switch his allocation. An initial endowment is provided for the buyers. In the Fisher case it is money and in the case of the Walrasian model it is a portfolio of goods (which may include money as a special case).

The existence of such equilibrium prices has been shown by Arrow and Debreau, under some mild assumptions. The proof is existential, however. Since then, there has been considerable interest in the computation of market equilibria in economic models. The utility of buyer i for the goods is given by $u_i(X_i)$ where X_i is the vector of allocation $\{x_{i1}, x_{i2} \dots x_{im}\}$. A number of utility functions have been used in this context, which include linear functions, the *Cobb-Douglas functions* of the form $u_i(x) = \prod_j (x_{ij})^{a_{ij}}$ for constants a_{ij} such that $\sum_j a_{ij} = 1$. Another class of functions which are useful is the *CES* function which is $u_i(x) = (\sum_j (c_{ij}x_{ij})^\rho)^{1/\rho}$ where $-\infty \leq \rho \leq 1, \rho \neq 0$. c_{ij} are constants.

The market equilibrium problem has been solved for a number of special cases using a variety of algorithmic techniques. The combinatorial techniques are : (a) primal-dual techniques algorithms based on maximum flows [4, 5] and (b) the auction based approaches [8] (c) Other classes of iterative procedures termed tâtonnement processes. Non-linear or convex programming techniques, which express the equilibrium problem as a convex programming problem, may be found in a variety of works starting from the works of Eisenberg and Gale [7, 6] in 1959 to the works (in Russian) of Primak et al. [11–13]. . Polynomial time approximation schemes which use a tâtonnement process was recently established in exchange economies with *weak gross substitutes* (WGS) utilities [3].

Auction based approaches have been shown to efficiently find (approximate) solutions to a wide class of problems [1]. In the context of the market equilibrium problem, auction based approaches are a subset of *tâtonnement* processes suggested in the economics literature, in fact by Walras himself. Such techniques may be very insightful in practice. It is very desirable to design markets where interaction of self-motivated trading agents provably leads to a market equilibrium in a reasonable amount of time. Auction based approaches may indeed help in designing such markets.

The auction based approach for the market equilibrium problem presented in [8] found an approximate solution for a market with linear utilities. Using price-rollback, the auction approach finds an exact solution to the problem [9]. Further, path auctions improve the complexity of the algorithm to the best known bound [9]. Auction algorithms has also been extended to non-linear case where the utilities are separable gross substitute [10]. In this paper we show that the basic auction mechanism of [10], can also find $(1+\epsilon)$ approximate market equilibrium for a larger class of utility function which includes CES, in the range that CES is WGS, and Cobb-Douglas utility functions. This is significant since this class includes functions widely used in economic models. The algorithm has a complexity which is a function of $O(1/\epsilon)$.

In Section 2 we define the market model and provide a characterization of utility functions. In Section 3 we outline our algorithm. The proof of correctness and complexity is similar to that in [10] and is skipped.

2 Market Model

Consider a market consisting of a set of n buyers and a set of m divisible goods. Buyer i has, initially, an amount of money equal to e_i. The amount of good j available in the market is a_j. Buyer i has a utility function, $U_i : R_+^M \to R_+$ which is non-decreasing, concave and differentiable in the range $0 \leq X_i \leq A$ where $A = (a_1, a_2, \ldots, a_m)$. Given prices $P = \{p_1, p_2, \ldots, p_m\}$ of these m goods, a buyer uses its money to purchase goods that maximize its total utility subject to its budget constraint. Thus a buyer i will choose an allocation $X_i \equiv (x_{i1}, x_{i2}, \ldots, x_{im})$ that solves the following buyer program $B_i(P)$:

$$\text{Maximize} : U_i(X_i) \tag{1}$$

$$\text{Subject to:} \sum_{1\leq j\leq m} x_{ij} p_j \leq e_i \tag{2}$$

and $\forall j : \ x_{ij} \geq 0$.

We say that the pair (X, P), $X = (X_1, X_2 \ldots X_n)$ forms a market equilibrium if (a) the vector $X_i \in R_+^n$ solves the problem $B_i(P)$ for all users i and (b) there is neither a surplus or a deficiency of any good i.e., $\forall j : \sum_{1\leq i\leq n} x_{ij} = a_j$.

The prices P are called market clearing prices and the allocation X is called an equilibrium allocation at price P. Let $v_{ij} : R_+^m \to R_+$ be equal to $\frac{\partial U_i(X_i)}{\partial x_{ij}}$. Since U_i is assumed to be differentiable for all i, v_{ij} is well defined for all i, j.

Using the theory of duality it can be shown that the optimal solution X_i to $B_i(P)$ will satisfy the following:

$$\forall i : \sum_{1 \le j \le m} x_{ij} p_j = e_i \tag{3}$$

$$\forall j : \alpha_i p_j \ge v_{ij}(X_i) \tag{4}$$

$$\forall j : x_{ij} > 0 \Rightarrow \alpha_i p_j = v_{ij}(X_i) \tag{5}$$

and $\forall i : \alpha_i \ge 0, \forall i, j : x_{ij} \ge 0$. The equations (3) imply that all the buyers have exhausted their budget. Equations (4) and (5) imply that (a) that every buyer has the same marginal utility per unit price on the goods it gets and (b) every good that a buyer is not allocated provides less marginal utility.

2.1 Uniformly Separable Utilities

We say that a utility function U_i is *uniformly separable* iff $v_{ij} \equiv \frac{\partial U_i(X_i)}{\partial x_{ij}}$ can be factored as: $v_{ij}(X_i) = f_{ij}(x_{ij}) g_i(X_i)$ such that f_{ij} is a strictly decreasing function. The following utility functions can be verified to be uniformly separable and gross substitute:

- CES (constant elasticity of substitution $u(X_i) = (\sum_i (w_{ij} x_{ij})^\rho)^{1/\rho}$, where $0 < \rho < 1$;
- Cobb-Douglas utility $u(X_i) = \Pi_j (x_{ij})^{a_{ij}}$ where $a_{ij} \ge 0$ and $\sum_j a_{ij} = 1$.

A buyer is said to have *gross substitute* demand for goods iff increasing the price of a good does not decrease the buyer's demand for other goods. Similarly, an economy is said to have gross substitutes demand iff increasing the price of a good does not decrease the total demand of other goods. Clearly, if every buyer has gross substitute demand then so does the economy. The following result characterizes the class of uniformly separable concave gross substitute utility functions.

Lemma 1. *Let U_i be a concave, strictly monotone, uniformly separable function ($\frac{\partial U_i(X_i)}{\partial x_{ij}} = f_{ij}(x_{ij}) g_i(X_i)$). U_i is gross substitute iff for all j, $y f_{ij}(y)$ is a non-decreasing function of the scalar y.*

Proof. We first prove that if U_i is a gross substitute function then $y f_{ij}(y)$ is non-decreasing. Assume, for contradiction, that there are scalars y and y' such that $y' < y$ and $y' f_{ij}(y') > y f_{ij}(y)$. Choose a price P and an optimal solution X_i of $B_i(P)$ such that $x_{ij} = y$ (it is always possible to do so because of strict monotonicity of U_i). Let α_i be the optimal dual solution of $B_i(P)$. The optimality conditions (4) and (5) for the dual of the program $B_i(P)$ can be rewritten as:

$$\forall j : x_{ij} > 0 \Rightarrow f_{ij}(x_{ij}) g_i(X_i) = \alpha_i p_j \tag{6}$$

$$\forall j : \alpha_i p_j \ge f_{ij}(x_{ij}) g_i(X_i) \tag{7}$$

Construct a corresponding (P', X'_i, α'_i) such that $x'_{ik} = x_{ik}$, $p'_k = p_k$ for all $k \neq j$, $x'_{ij} = y'$, $p'_j = p_j f_{ij}(x'_{ij})/f_{ij}(x_{ij}) = p_j f_{ij}(y')/f_{ij}(y)$ and $\alpha'_i = \alpha_i g_i(X'_i)/g_i(X_i)$. Note that the solution (X'_i, α'_i, P') satisfies (6) and (7). Now,

$$x'_{ij}p'_j = y'p'_j = p_j y' f_{ij}(y')/f_{ij}(y) > p_j y f_{ij}(y)/f_{ij}(y) = yp_j = x_{ij}p_j$$

Thus, $\sum_{j=1}^m x'_{ij}p'_j > \sum_{j=1}^m x_{ij}p_j = e_i$, implying that the solution X'_i violates the optimality condition (3) of program $B_i(P')$. Therefore, the optimal solution (X''_i, α''_i) of $B_i(P')$ must have $x''_{ij} < x'_{ij}$ for some j. Since (X'_i, α'_i) and (X''_i, α''_i) satisfy (6) and (7) for the same price P' and f_{ij} is strictly decreasing for all j, we must also have $x''_{ij} < x'_{ij}$ for all j. From the definition of X'_i, it is clear that this violates the gross substitutability condition.

We next show that if $yf_{ij}(y)$ is non-decreasing then the goods satisfy gross-substitutability. Consider an optimal solution X_i of $B_i(P)$. If $x_{ij} > 0$, equation (6) gives $x_{ij}p_j = x_{ij}f_{ij}(x_{ij})g_i(X_i)/\alpha_i$. Consider $P' > P$. For this price vector we construct a feasible solution X'_i satisfying (6) and (7) as follows: If $p'_j = p_j$ then $x'_{ij} = x_{ij}, \forall i$. Alternately, if $f_{ij}(0)g_i(X_i) < \alpha_i p'_j$ then set x'_{ij} to zero, else choose x'_{ij} such that $f_{ij}(x'_{ij})g_i(X_i) = \alpha_i p'_j$. Set $\alpha'_i = \alpha_i(g_i(X'_i)/g_i(X_i))$. By definition, the solution X'_i satisfies the complementary slackness conditions (6). Since $P' > P$, X'_i also satisfies (7). Also since f_{ij} is a strictly decreasing function $p'_j > p_j \Rightarrow x'_{ij} < x_{ij}$. Now, if $x'_{ij} > 0$ then

$$x'_{ij}p'_j = x'_{ij}f_{ij}(x'_{ij})g_i(X'_i)/\alpha'_i \leq x_{ij}f_{ij}(x_{ij})g_i(X_i)/\alpha_i = x_{ij}p_j$$

If $x'_{ij} = 0$ then also we have $x'_{ij}p'_j \leq x_{ij}p_j$ The above equations give

$$\sum_{j=1}^m x'_{ij}p'_j \leq \sum_{j=1}^m x_{ij}p_j = e_i$$

Therefore, any optimal solution (X''_i, α''_i) of the program $B_i(P')$ should have $x''_{ij} > x'_{ij}$ for some j. Since (X''_i, α''_i) and (X'_i, α'_i) both satisfy (6) and (7) for the same price P' and f_{ij} is strictly decreasing, we must have $x''_{ij} \geq x'_{ij}$ for all j. Gross substitutability now follows from the definition of X'_i.

3 An Auction Algorithm for Market Clearing

An auction algorithm similar to that in [10] solves the market equilibrium problem for the uniformly separable gross substitute utility functions. For the sake of completeness we give a brief description of the algorithm.

The algorithm (formally presented in Figure 1) begins with assigning all the goods to one buyer (say buyer 1) and adjusting the prices such that (a) all the money of the buyer is exhausted and (b) the initial allocation is optimal for the buyer. During the course of the algorithm, goods may be allocated at two prices, p_j and $p_j/(1+\epsilon)$. The allocation of good j to buyer i at price p_j is represented by h_{ij} and the allocation at price $p_{ij}/(1+\epsilon)$ is represented by y_{ij}. The total allocation of good j to buyer i is given by $x_{ij} = h_{ij} + y_{ij}$. Define

```
algorithm main
   initialize
   while ∃i : r_i > εe_i
      while (r_i > 0) and
        (∃j : α_ij p_j < f_ij(x_ij) g_i(X_i))
        if ∃k : y_kj > 0 then
           outbid(i, k, j, α_ij)
        else raise_price(j)
      end while
      j = arg max_l α_il
      if ∃k : y_kj > 0
        outbid(i, k, j, α_ij/(1 + ε))
        α_ij = f_ij(x_ij) g_i(X_i)/p_j
      else raise_price(j)
   end while
end algorithm main

procedure raise_price(j)
   ∀i : y_ij = h_ij; h_ij = 0;
   p_j = (1 + ε) p_j
end procedure raise_price

procedure initialize
   ∀i, ∀j : h_ij = 0
   ∀i ≠ 1, ∀j : y_ij = 0
   ∀j : y_1j = a_j ;  a = (a_1, ..., a_n);
   ∀j : α_1j = (Σ_j a_j f_1j(a_j) g_1(a))/e_i
   ∀j : p_j = f_1j(a_j) g_1(a)/α_1
   ∀i ≠ 1 : α_i = f_ij(0) g_i(0̂)/p_j ;  r_i = e_i
   ∀i ≠ 1, ∀j : α_ij = f_ij(x_ij) g_i(X_i)/p_j
   r_1 = 0
end procedure initialize

procedure outbid(i, k, j, α)
   t_1 = y_kj
   t_2 = r_i/p_j
   if (f_ij(a_j) g_i(X_i) ≥ α p_j) then
      t_3 = a_j  (x'_ik = x_ik, k ≠ j;  x'_ij = a_j)
   else
      t_3 = min δ : f_ij(x_ij + δ) g_i(X_i) = α p_j
   t = min(t_1, t_2, t_3)
   h_ij = h_ij + t
   r_i = r_i − t p_j
   y_kj = y_kj − t
   r_k = r_k + t p_j/(1 + ε)
end procedure outbid
```

Fig. 1. The auction algorithm

the surplus of a buyer i as $r_i = \sum_{j=1}^{m}(h_{ij}p_j + y_{ij}p_j/(1+\epsilon))$. Define the total surplus in the system as $r = \sum_{i=1}^{n} r_i$. The parameter ϵ is called the minimum bid increment and determines the accuracy of the final solution obtained. Now buyers with unspent money try to acquire items that give them the maximum utility per unit money, by outbidding other buyers and raising the price of items. The bidding is carried out till all the buyers have little unspent money.

Note this algorithm is very characteristic of a typical auction market. The bidding is asynchronous, decentralized and local. The buyers do not have to coordinate their actions. Any buyer with surplus money can place a bid on an item that maximizes the value of the buyer, this outbidding other buyers. The process stops when the unspent money with every buyer is sufficiently small.

To show convergence of the algorithm, the bidding may be organized in rounds. In each round every buyer (i) is picked once and reduces his surplus to 0, i.e. $r_i = 0$. Now it can be shown that in every round of bidding, the total unspent money decreases by a factor of $(1+\epsilon)$. This gives the following bound on the time complexity of the algorithm ($v_{max} = \max_{ij} v_{ij}(0)$) (see [10]).

Theorem 1. *The auction algorithm terminates in* $O((E/\epsilon)\log((evv_{max})/(\epsilon e_{min}v_{min}))\log n)$ *steps,*

4 Conclusions

Naive auction algorithms give approximate market equilibrium. The approximation is related to the minimum bid increment parameter ϵ used in the algorithm. It was shown that for linear utility functions, auctions with suitable price-rollbacks and modifications to ϵ lead to exact market equilibrium. It will be interesting to see if any such approach may also work for the general class of gross substitute utilities.

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