Fuzzy Logic

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Boolean (Crisp) Logic

- Is it raining
  - Yes/No
  - True/False

Crisp answer

If answer – may be, may not be, … (vague)

Not a crisp answer (fuzzy answer)

Fuzzy Logic

- Is this person tall?
  - Yes/No
  - Very tall, quite tall, not so tall, not tall,

7’ 6’ 5’6” 5’

Better (more precise) description !

Introduced by Zadeh (1965), University of California Berkley

Fuzzy Logic

- Approximation (“granulation”)

A color can be described precisely using RGB values, or it can be approximately described as “red”, “blue”, etc.

- Degree (“graduation”)

Two different colors may both be described as “red”, but one is considered to be more red than the other

- Fuzzy logic attempts to reflect the human way of thinking

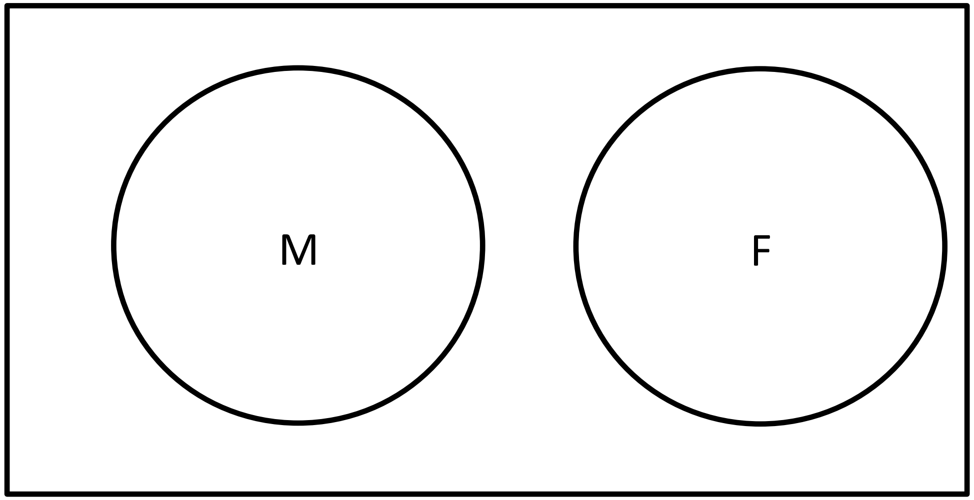
Fuzzy Set

X = Entire population in a class

M = All male population (m_1, m_2, m_3, …, m_N)

F = All female population (f_1, f_2, f_3, …, f_L)

Crisp Sets



Fuzzy Set

X = Entire population in a class of SIV895/CSM802

S = All good students

S = { (s, g) | s ε X } and g(s) is a measurement of goodness of student s.

For example:

S = { (Rakesh, 0.8), (Sunita, 0.7), (Farhan, 0.1), (Joseph, 0.9) } etc

Fuzzy Set

Crisp Set	Fuzzy Set
$S = \{ s \mid s \varepsilon \ X \}$	$F = \{ (s, \mu) \mid s \varepsilon \ X \}$ $\mu(s)$ is the degree of s.
It is a collection of elements	It is a collection of ordered pairs
Inclusion is crisp (yes or no)	Inclusion is fuzzy, i.e. if present then with a degree of membership

Crisp set is a fuzzy set with extreme membership values (0 or 1).

Related Terms

Fuzzy relation

Relationships can also be expressed on a scale of 0 to 1 e.g. degree of *resemblance* between two people

Fuzzy variable

Variable with (labels of) fuzzy sets as its values

Linguistic variable

Fuzzy variable with values that are words or sentences in a language e.g. variable *color* with values *red*, *blue*, *yellow*, *green*…

Linguistic hedge

Term used as a modifier for basic terms in linguistic values e.g. words such as *very*, *a bit*, *rather*, *somewhat*, etc.

Fuzzy Set

Examples Fuzzy Set

If *cold* is a fuzzy set, exact temperature values might be mapped to the fuzzy set as follows:

- 15 degrees → 0.2 (slightly cold)
- 10 degrees → 0.5 (quite cold)
- 0 degrees → 1 (extremely cold)

Membership Function

If X is a universe of discourse and x ε X, then a fuzzy set A in X is defined as a set of ordered pairs, that is $A = \{ (x, \mu_A(x)) | x \varepsilon X \}$ where, $\mu_A(x)$ is called the membership function for the fuzzy set A.

Note: $\mu_A(x)$ map each element of X onto a membership grade (or membership value) between 0 and 1 (both inclusive).

Membership Function

A fuzzy set is completely characterized by its membership function. So, it would be important to learn how a membership function can be expressed (mathematically or otherwise).

Note: A membership function can be on
(a) a discrete universe of discourse and
(b) a continuous universe of discourse.

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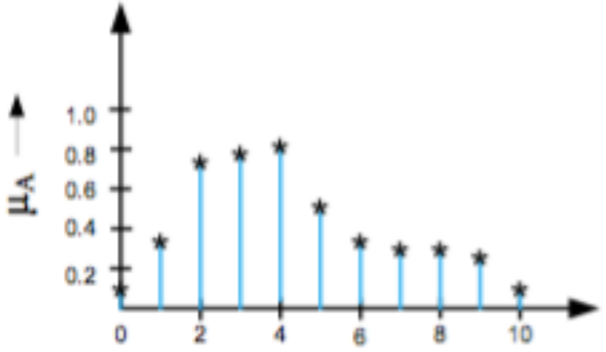
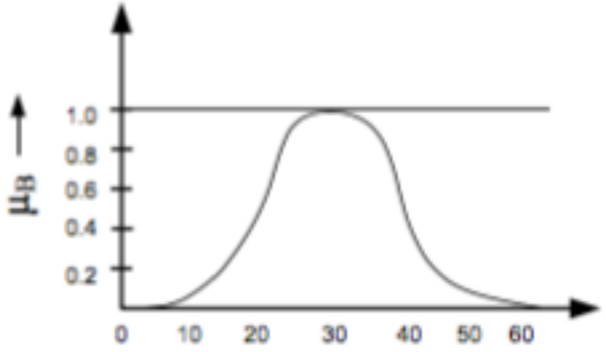
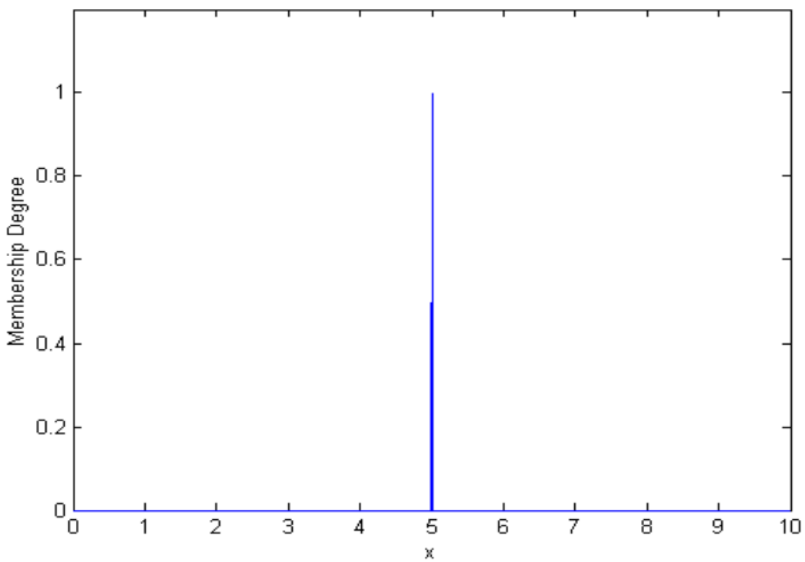




Membership Function

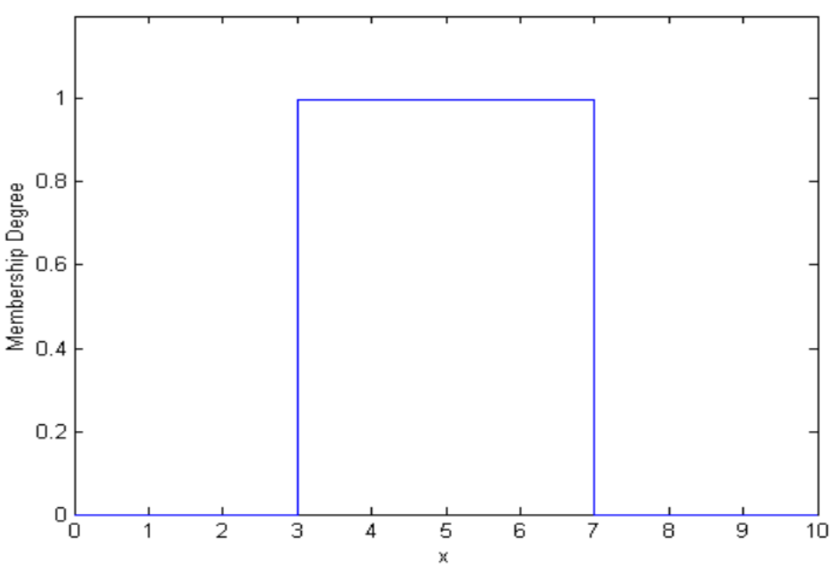
Singleton MF



$$\mu(x) = \begin{cases} 1, & x = c \\ 0, & \text{otherwise} \end{cases}$$

Membership Function

Rectangular MF

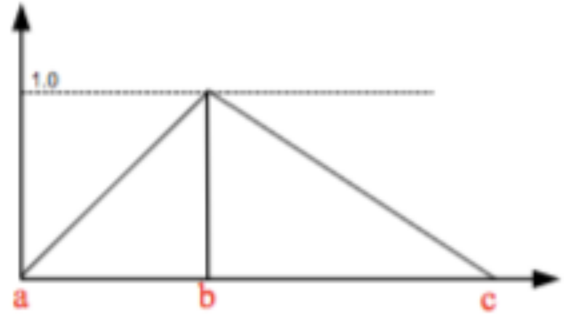


$$\mu(x) = \begin{cases} 1, & l \leq x \leq r \\ 0, & \text{otherwise} \end{cases}$$

Membership Function

Triangular MF

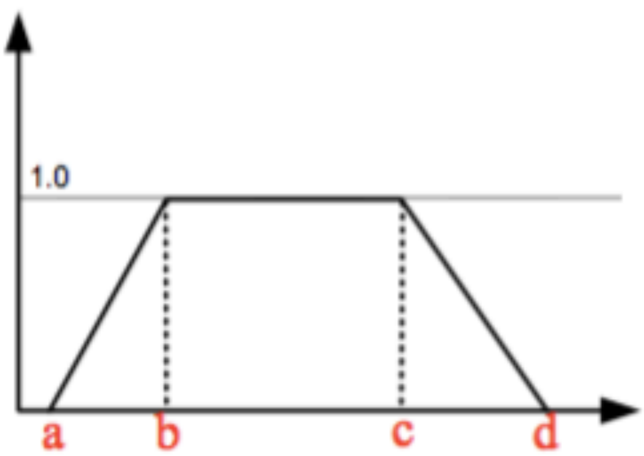
$$triangle(x;a,b,c) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a \le x \le b \\ \frac{c-x}{c-b} & \text{if } b \le x \le c \\ 0 & \text{if } c \le x \end{cases}$$



Membership Function

Trapeziodal MF

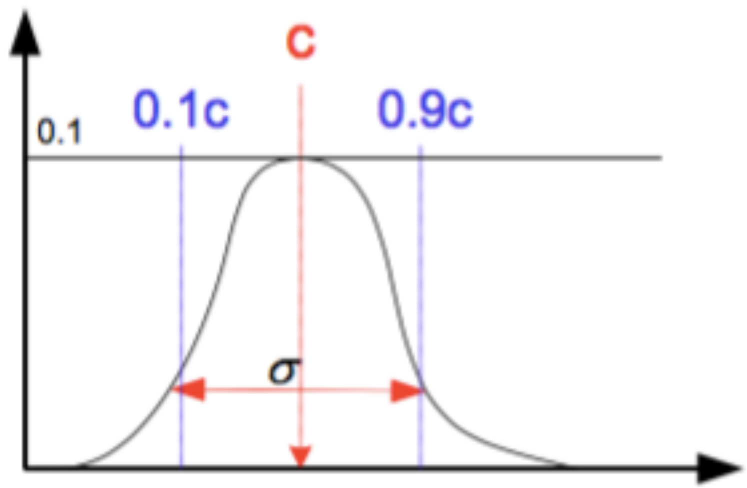
$$trapeziod(x; a, b, c, d) = \begin{cases} 0 & \text{if } x \leq a \\ \frac{x-a}{b-a} & \text{if } a \leq x \leq b \\ 1 & \text{if } b \leq x \leq c \\ \frac{d-x}{d-c} & \text{if } c \leq x \leq d \\ 0 & \text{if } d \leq x \end{cases}$$



Membership Function

Gaussian MF

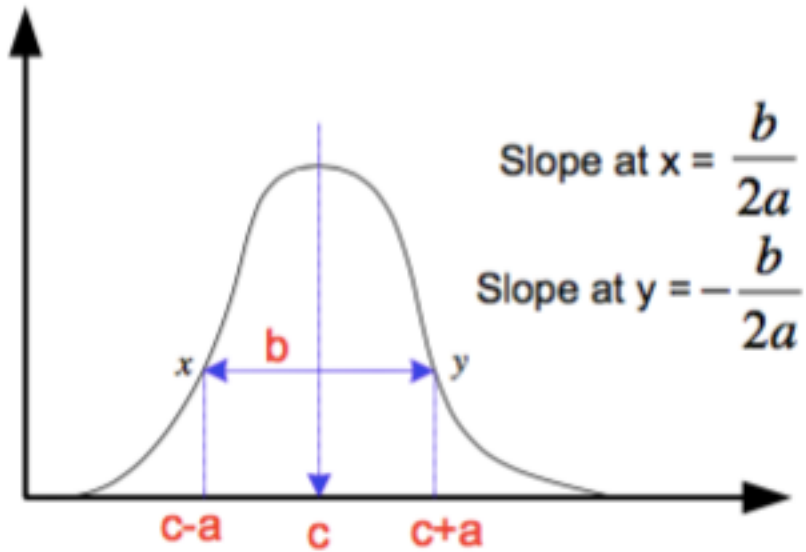
$$\text{gaussian}(x;c,\sigma) = e^{-\frac{1}{2}\left(\frac{x-c}{\sigma}\right)^2}.$$



Membership Function

Bell (Cauchy) MF

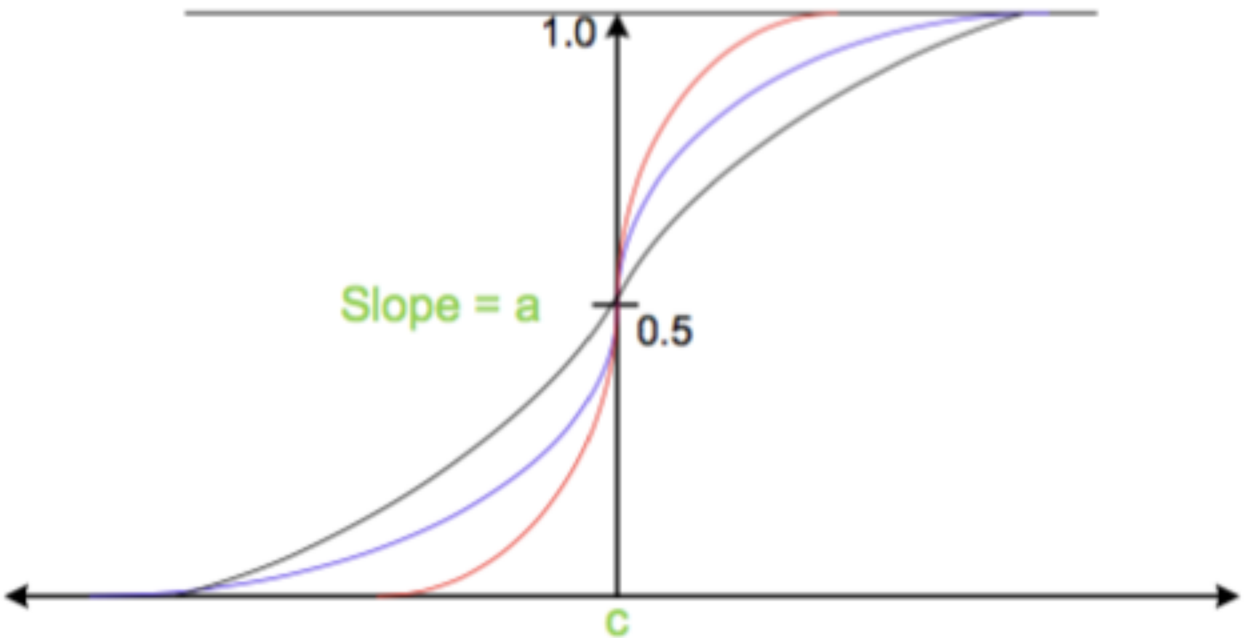
$$\text{bell}(x; a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$



Membership Function

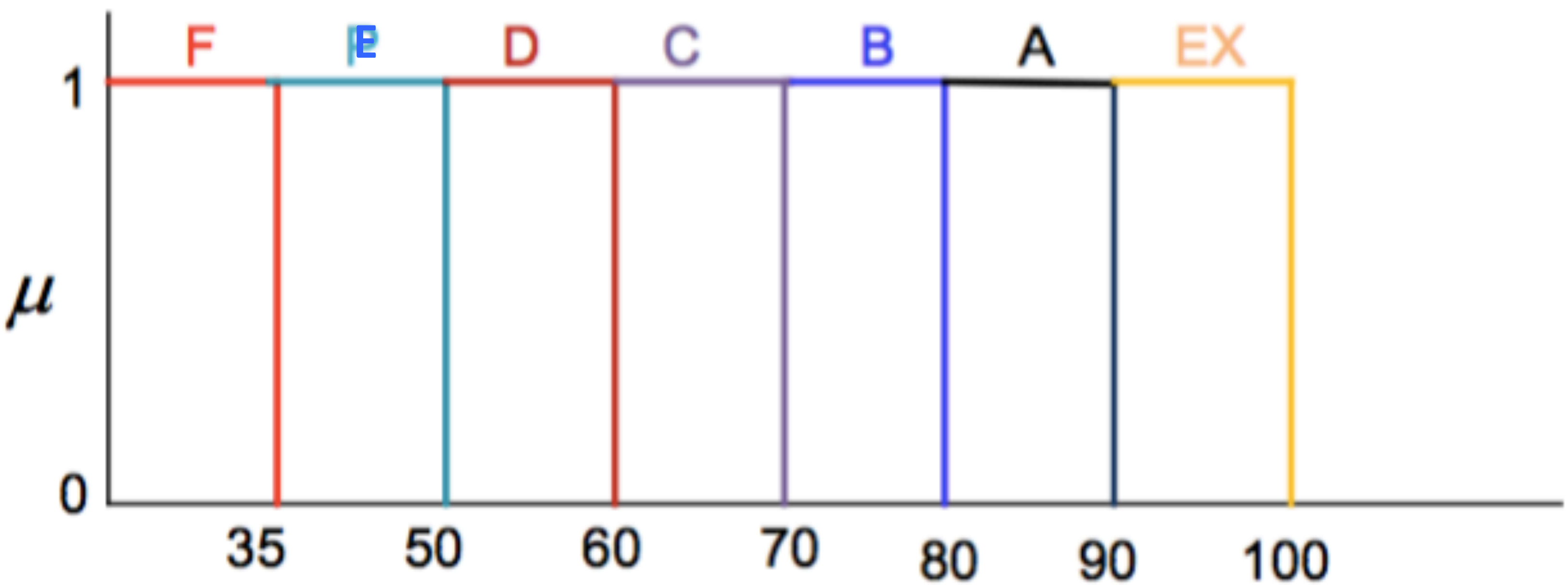
Sigmoidal MF

$$\text{Sigmoid}(x;a,c) = \frac{1}{1+e^{-\left[\frac{a}{x-c}\right]}}$$



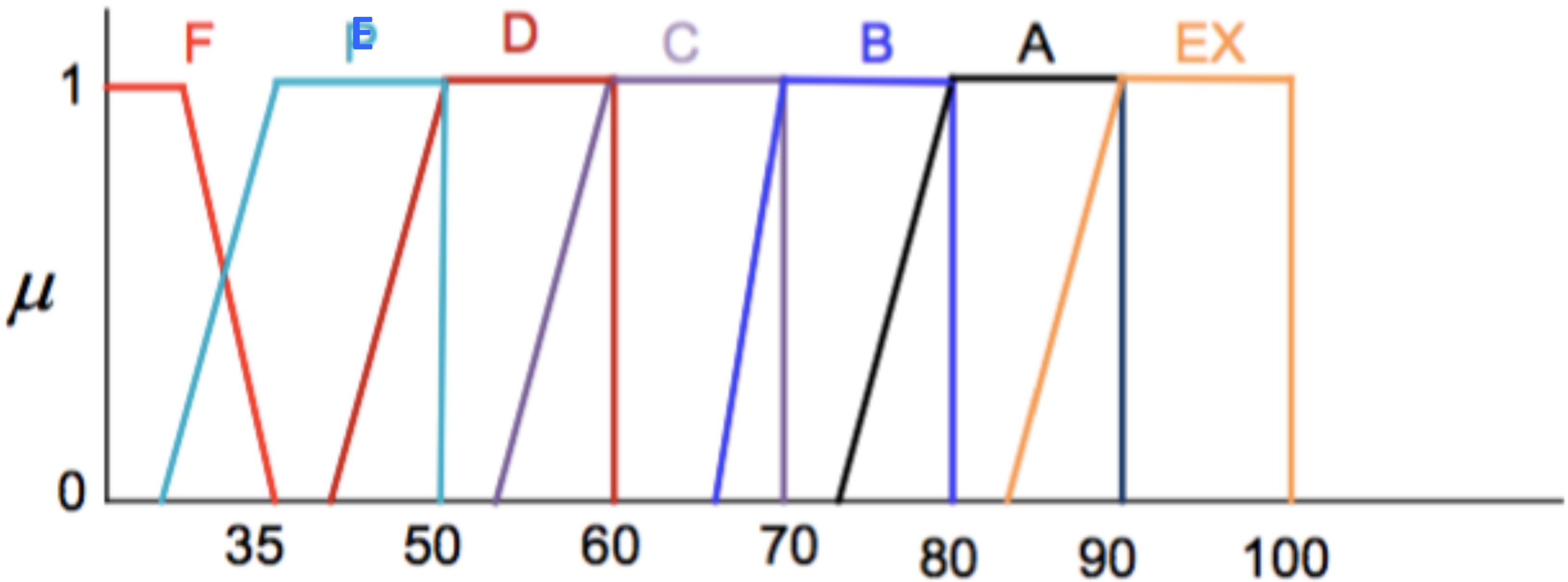
Example

Course Grading (Crisp)



Example

Course Grading (Fuzzy)

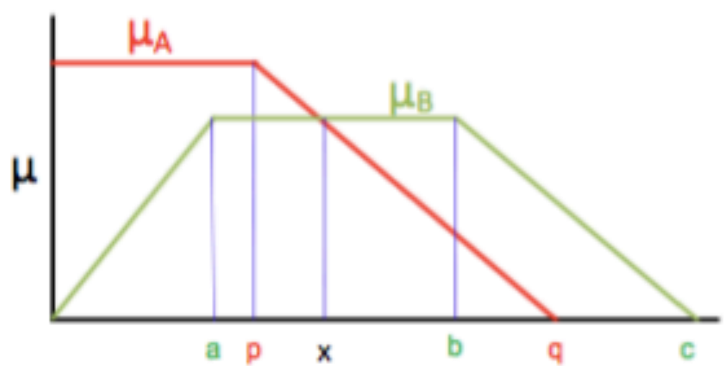


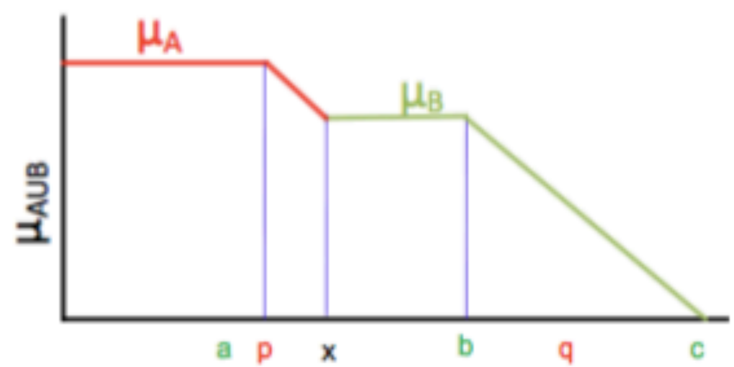
Operations on Fuzzy Sets

Union ($A \cup B$):

$$\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cup B = \{(x_1, 0.5), (x_2, 0.3), (x_3, 0.5)\}$



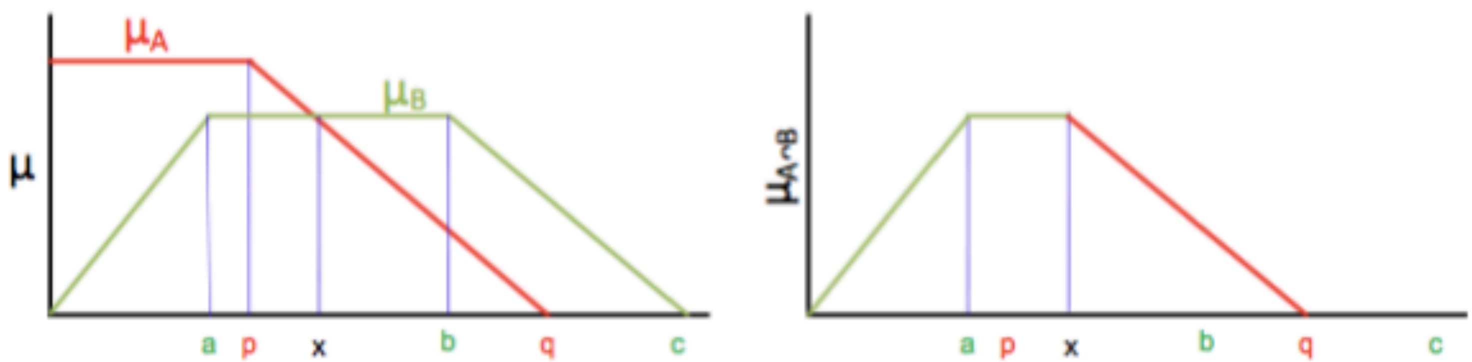


Operations on Fuzzy Sets

Intersection ($A \cap B$):

$$\mu_{A\cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}$$

Example:
$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$ and
$B = \{(x_1, 0.2), (x_2, 0.3), (x_3, 0.5)\}$;
$C = A \cap B = \{(x_1, 0.2), (x_2, 0.1), (x_3, 0.4)\}$



Operations on Fuzzy Sets

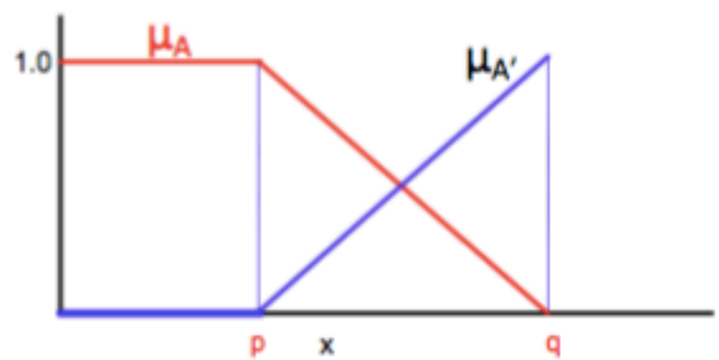
Complement (A^C):

$$\mu_{A_{A^C}}(x) = 1 - \mu_A(x)$$

Example:

$A = \{(x_1, 0.5), (x_2, 0.1), (x_3, 0.4)\}$

$C = A^C = \{(x_1, 0.5), (x_2, 0.9), (x_3, 0.6)\}$



Properties

Commutativity :

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

Associativity :

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributivity :

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

Properties

Idempotence :

$$A \cup A = A$$
$$A \cap A = A$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$

Transitivity :

$$\text{If } A \subseteq B, B \subseteq C \text{ then } A \subseteq C$$

Involution :

$$(A^c)^c = A$$

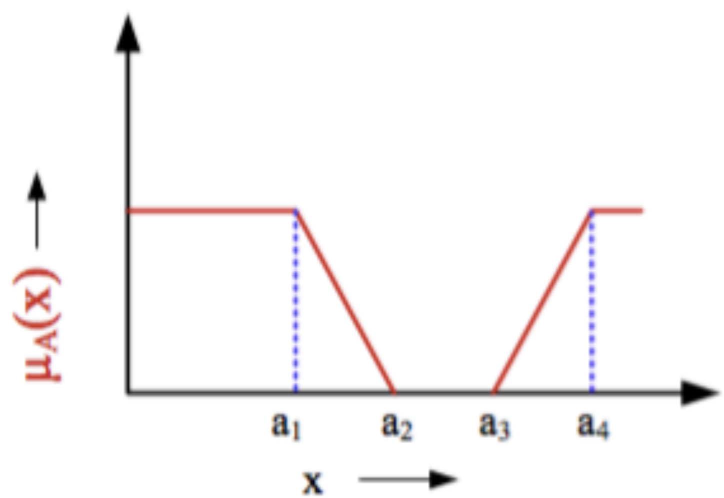
De Morgan's law :

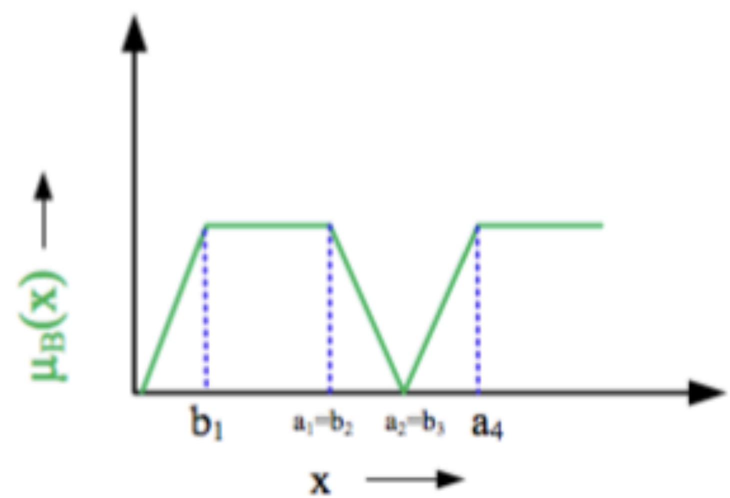
$$(A \cap B)^c = A^c \cup B^c$$
$$(A \cup B)^c = A^c \cap B^c$$

Operations on Fuzzy Sets

Example

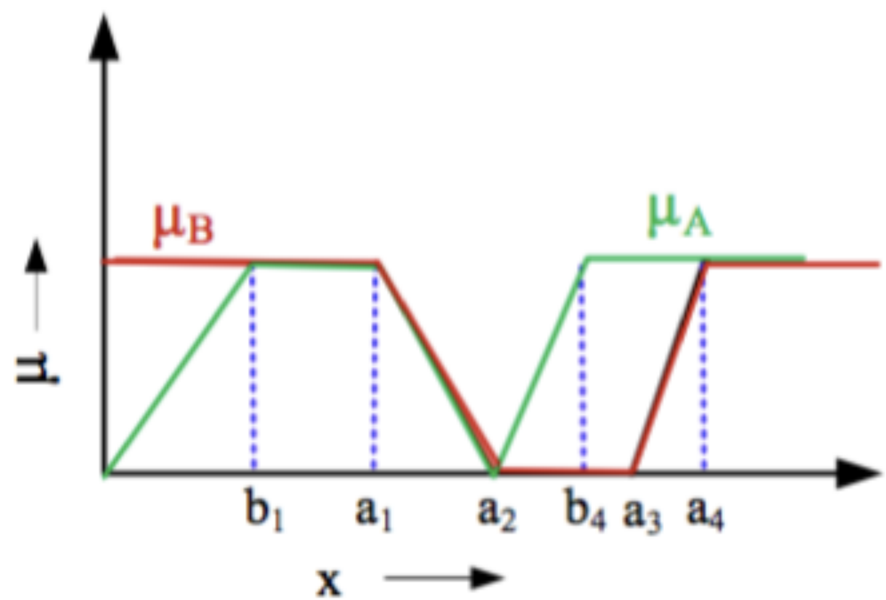
Let A and B are two fuzzy sets defined over a universe of discourse X with membership functions $\mu_A(x)$ and $\mu_B(x)$, respectively. Two MFs $\mu_A(x)$ and $\mu_B(x)$ are shown graphically.





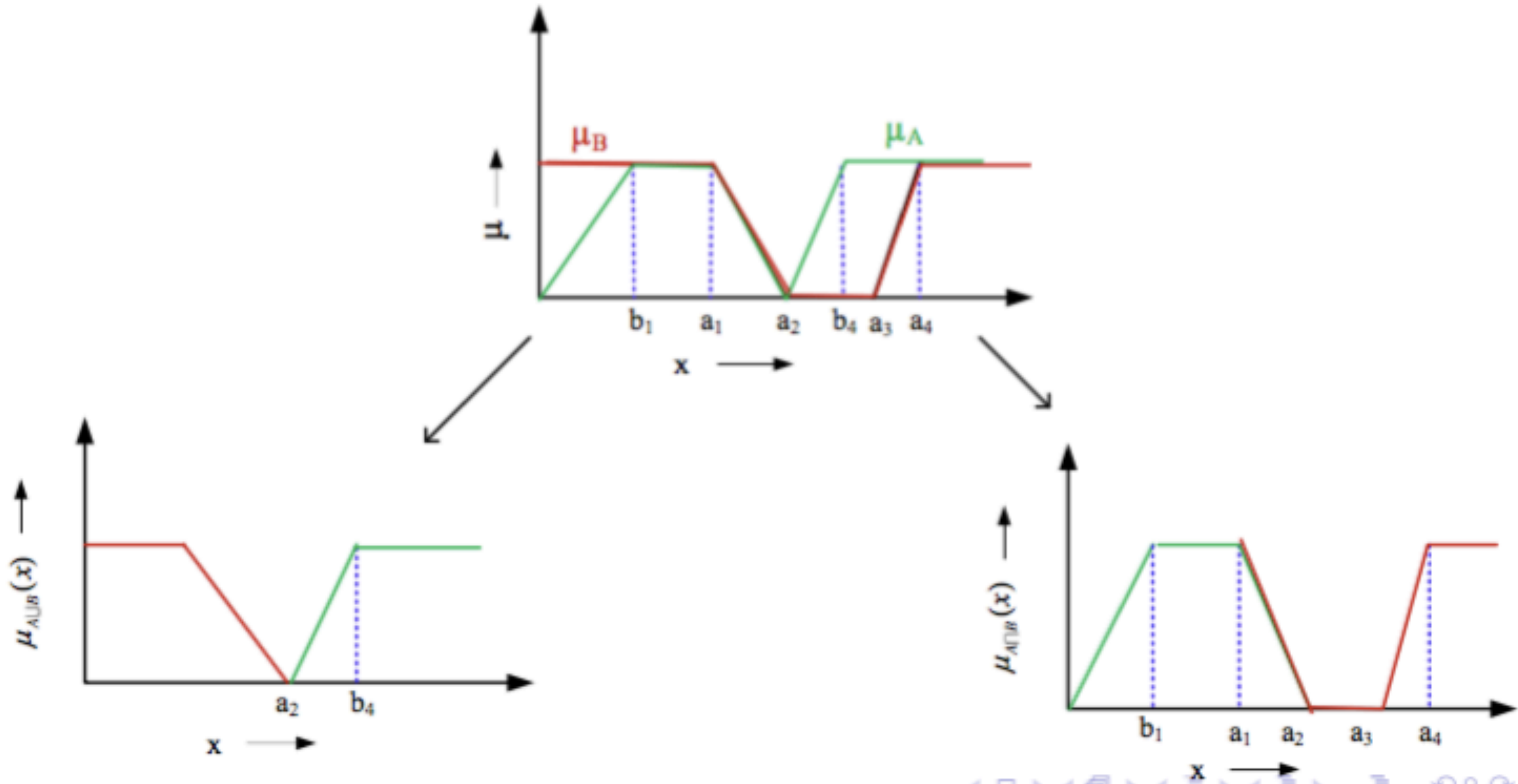
Operations on Fuzzy Sets

Example



Operations on Fuzzy Sets

Example

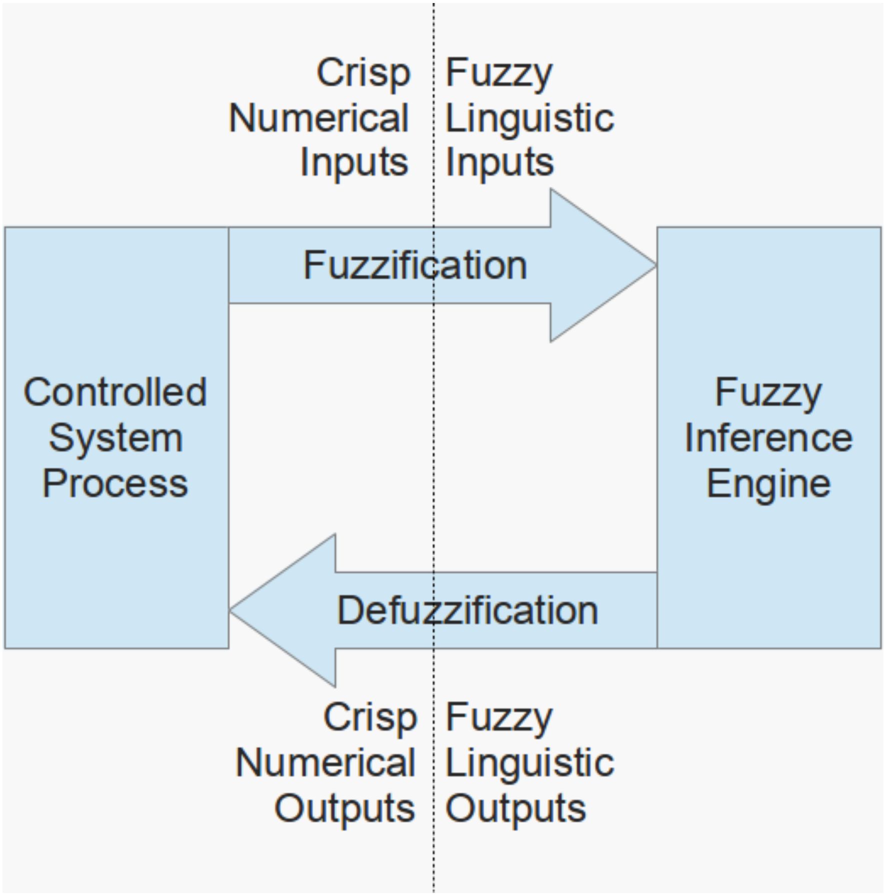


Fuzzy vs Probability

Fuzzy : When we say about certainty of a thing
Example: A patient come to the doctor and he has to diagnose so that medicine can be prescribed. Doctor prescribed a medicine with certainty 60% that the patient is suffering from flue. So, the disease will be cured with certainty of 60% and uncertainty 40%. Here, in stead of flue, other diseases with some other certainties may be.

Probability: When we say about the chance of an event to occur
Example: India will win the T20 tournament with a chance 60% means that out of 100 matches, India own 60 matches.

Fuzzy System



Operations on Fuzzy Sets

Exercise

Consider the following two fuzzy sets A and B defined over a universe of discourse [0,5] of real numbers with their membership functions

$$\mu_A(x) = \frac{x}{1+x} \text{ and } \mu_B(x) = 2^{-x}$$

$A \cup B$

. $A \cap B$