Curves

Polylines: Piecewise linear approximation to curves

Not Smooth

Curves

High degree approximation

- Explicit

  $y=f(x)$

- Implicit

  $f(x,y)=0$

- Parametric

  $x=x(t),\ y=y(t)$

Curves

Explicit Representation

- $y = f(x)$
- Essentially a function plot over some interval
  $x \in [a,b]$
- Simple to compute and plot
- Simple to check whether point lies on curve
- Cannot represent closed and multi-value curves

Curves

Implicit Representation

- Define curves implicitly as solution of equation system

◦ Straight line in 2D: $\boldsymbol{ax + by + c} = 0$

◦ Circle of radius R in 2D: $\boldsymbol{x}^2 + \boldsymbol{y}^2 - \boldsymbol{R}^2 = 0$

◦ Conic Section: $\boldsymbol{Ax}^2 + 2\boldsymbol{Bxy} + \boldsymbol{Cy}^2 + 2\boldsymbol{Dx} + 2\boldsymbol{Ey} + \boldsymbol{F} = 0$

- Simple to check whether point lies on curve
- Can represent closed and multi-value curves

Curves

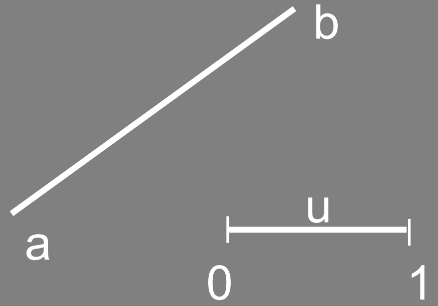
Parametric Curves

- Describe position on the curve by a parameter

$$u \in \mathbf{R}$$

$$\Rightarrow \text{2D Curve } \mathbf{c}(\mathbf{u}) = \{\mathbf{x}(\mathbf{u}), \mathbf{y}(\mathbf{u})\}$$

$$\text{e.g.,line} : \mathbf{c}(\mathbf{u}) = (1 - \mathbf{u})\mathbf{a} + \mathbf{u}\mathbf{b}$$



- Hard to check whether point lies on curve
- Simple to render
- Can represent closed and multi-value curves

Curves

Parametric Curves

Parametric curves form a rich variety of free form smooth curves

They are modeled as piecewise polynomials and have two aspects:

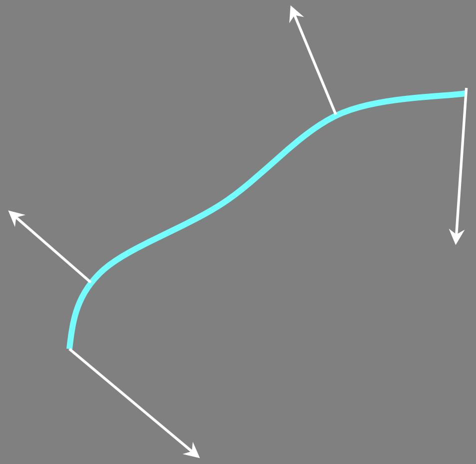
- Interpolation
- Approximation

"Splines"

Curves

Parametric Curves

Splines



Curves

Parametric Curves

Cubic Splines

$$P(t) = B_1 + B_2 t + B_3 t^2 + B_4 t^3 \qquad t_1 \le t \le t_2$$

$$= \sum_{i=1}^{4} B_i t^{i-1}$$

$$x(t) = \sum_{i=1}^{4} B_{ix} t^{i-1}$$

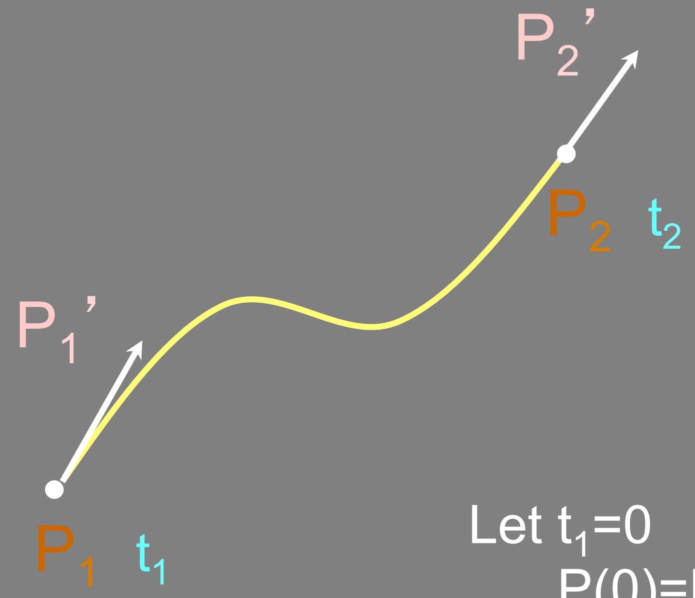
$$y(t) = \sum_{i=1}^{4} B_{iy} t^{i-1}$$

$$P'(t) = \sum_{i=1}^{4} B_i (i-1) t^{i-2} = B_2 + 2B_3 t^1 + 3B_4 t^2$$

Curves

Parametric Curves

Cubic Splines



Let $t_1=0$

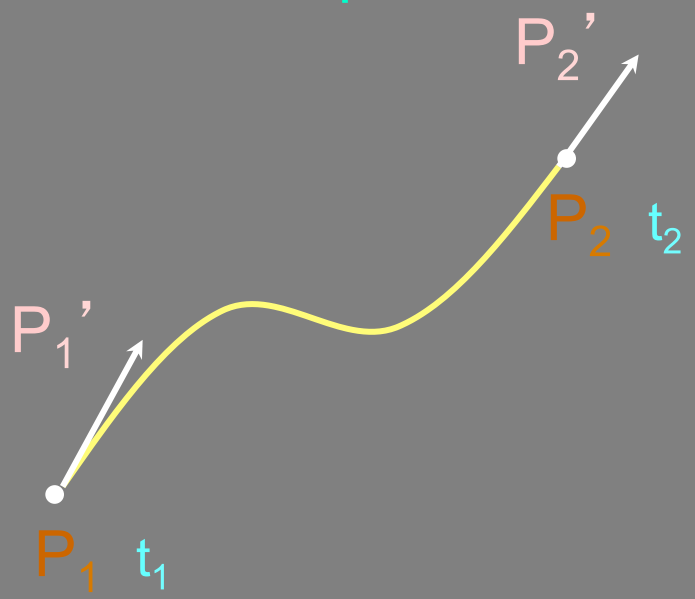
$P(0)=P_1$, $P(t_2)=P_2$

$P'(0)=P_1'$, $P'(t_2)=P_2'$

Curves

Parametric Curves

Cubic Splines



$$P(0) = B_1 = P_1$$

$$P(t_2) = \sum_{i=1}^{4} B_i t^{i-1} \Big|_{t=t_2}$$

$$= B_1 + B_2 t_2^{\ 1} + B_3 t_2^{\ 2} + B_4 t_2^{\ 3} = P_2$$

$$P'(0) = \sum_{i=1}^{4} B_i (i-1) t^{i-2} \Big|_{t=0} = B_2 = P_1'$$

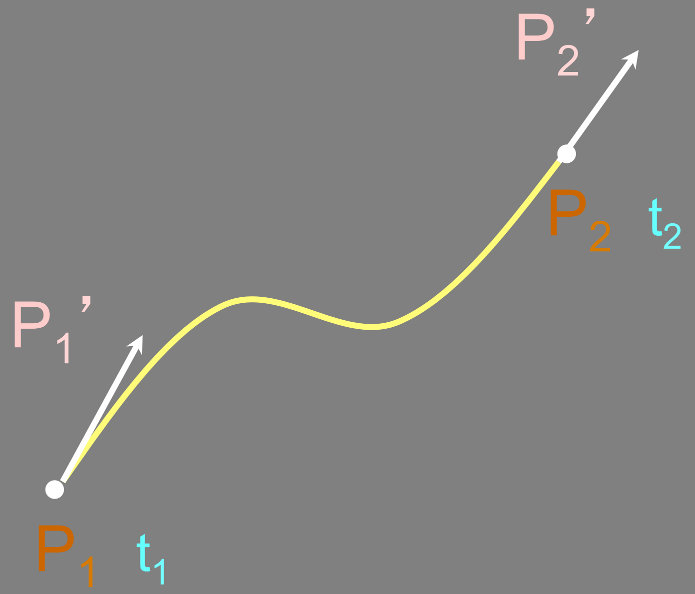
$$P'(t_2) = \sum_{i=1}^{4} B_i (i-1) t^{i-2} \Big|_{t=t_2}$$

$$= B_2 + 2B_3 t_2^{\ 1} + 3B_4 t_2^{\ 2} = P_2'$$

Curves

Parametric Curves

Cubic Splines



Solving for B_1 B_2 B_3 and B_4

$$B_1 = P_1,$$

$$B_2 = P_1',$$

$$B_3 = \frac{3(P_2 - P_1)}{t_2^2} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2}$$

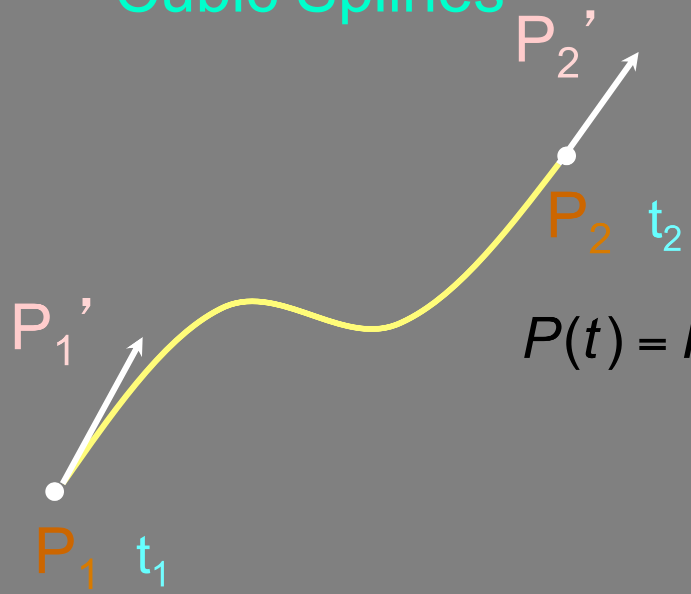
$$B_4 = \frac{2(P_1 - P_2)}{t_2^3} + \frac{P_1'}{t_2^2} + \frac{P_2'}{t_2^2}$$

Here P_1 and P_2 give the position of the endpoints and P_1' and P_2' give the direction of the tangent vectors.

Curves

Parametric Curves

Cubic Splines



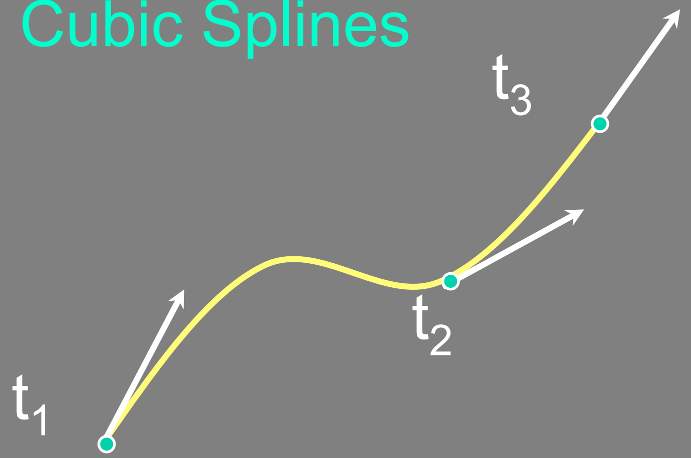
$$P(t) = P_1 + P_1' t + \left(\frac{3(P_2 - P_1)}{t_2^{\,2}} - \frac{2P_1'}{t_2} - \frac{P_2'}{t_2} \right) t^2 + \left(\frac{2(P_1 - P_2)}{t_2^{\,3}} + \frac{P_1'}{t_2^{\,2}} + \frac{P_2'}{t_2^{\,2}} \right) t^3$$

Thus, given the two end-points and the tangent vectors one can compute the cubic spline.

Curves

Parametric Curves

Cubic Splines



Joining of Segments

2 SEGMENTS: P_1 P_2 P_3 (Points)

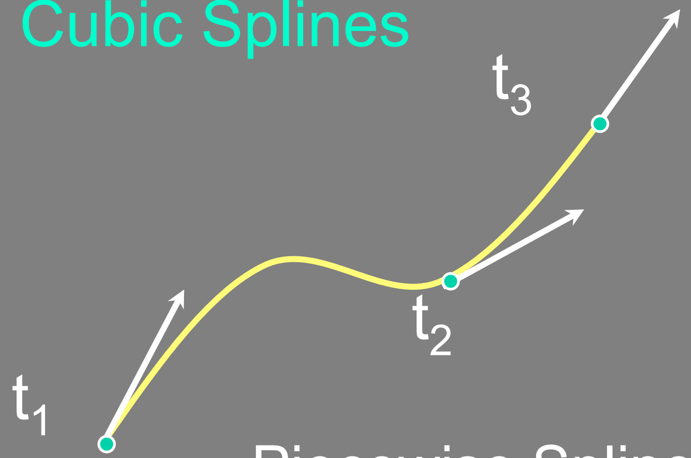
P_1' P_2' P_3' (Tangents)

P_2' is determined through the continuity condition

Curves

Parametric Curves

Cubic Splines



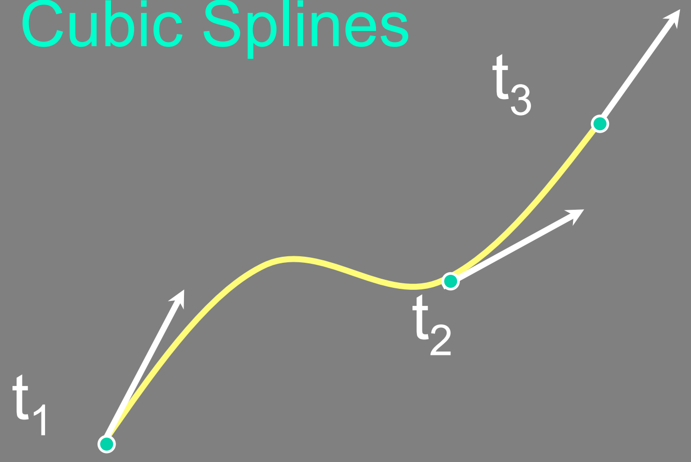
Piecewise Spline of degree ***k*** has continuity of order ***(k-1)*** at the internal joints.

Thus Cubic Splines have second order continuity i.e. $P_2''(t)$ is continuous over the joint

Curves

Parametric Curves

Cubic Splines



$$P''(t) = \sum_{i=1}^{4} (i-1)(i-2) B_i t^{i-3} \qquad t_1 \le t \le t_2$$

at $t = t_2$

First segment

$$P'' = 6B_4 t_2 + 2B_3$$

Second segment

$$P'' = 2B_3$$

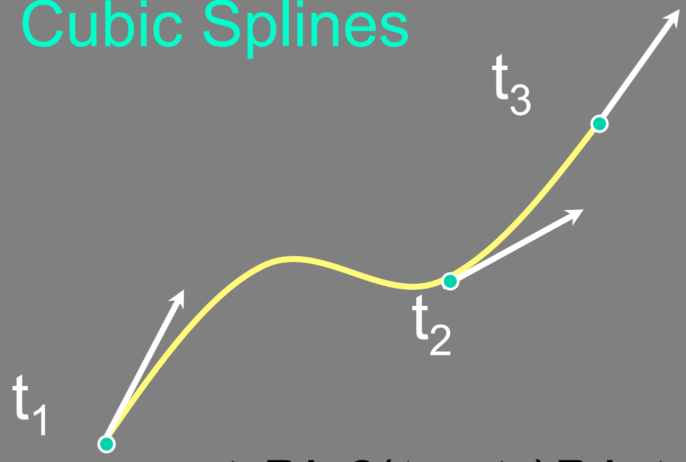
$$So, (6B_4 t_2 + 2B_3)_{seg1} = (2B_3)_{seg2}$$

Substitute the expressions for B_4 and B_3

Curves

Parametric Curves

Cubic Splines



$$t_3 P_1' + 2(t_3 + t_2) P_2' + t_2 P_3' = \frac{3}{t_2 t_3}\left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1)\right)$$

$$\begin{bmatrix} t_3 & 2(t_3 + t_2) & t_2 \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ P_3' \end{bmatrix} = \frac{3}{t_2 t_3}\left(t_2^2 (P_3 - P_2) + t_3^2 (P_2 - P_1)\right)$$

Curves

Parametric Curves

Cubic Splines

In general, for the ***k***th and ***(k+1)***th segment (1≤***k***≤n-2)

$$\begin{bmatrix} t_{k+2} & 2(t_{k+1}+t_{k+2}) & t_{k+1} \end{bmatrix} \begin{bmatrix} P_k' \\ P_{k+1}' \\ P_{k+2}' \end{bmatrix} = \frac{3}{t_{k+1}t_{k+2}}\left(t_{k+1}^2 (P_{k+2} - P_{k+1}) + t_{k+2}^2 (P_{k+1} - P_k) \right)$$

Set of n-2 equations form a linear system for the tangent vectors P_k'

Curves

Parametric Curves

Cubic Splines

$$\begin{bmatrix} t_3 & 2(t_2+t_3) & t_2 & 0 & \cdots \\ 0 & t_4 & 2(t_3+t_4) & t_3 & \\ \vdots & & \ddots & & \vdots \\ & \cdots\cdots\cdots & t_n & 2(t_n+t_{n-1}) & t_{n-1} \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ P_n' \end{bmatrix}$$

$$= \begin{bmatrix} \frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2)+t_3^2(P_2-P_1)\right) \\ \frac{3}{t_3 t_4}\left(t_3^2(P_4-P_3)+t_4^2(P_3-P_2)\right) \\ \vdots \\ \frac{3}{t_{n-1} t_n}\left(t_{n-1}^2(P_n-P_{n-1})+t_n^2(P_{n-1}-P_{n-2})\right) \end{bmatrix}$$

Curves

Parametric Curves

Cubic Splines

$$
\begin{bmatrix}
1 & 0 & \cdots\cdots & & & \\
t_3 & 2(t_2+t_3) & t_2 & & & \\
\vdots & t_4 & 2(t_3+t_4) & t_3 & & \vdots \\
\vdots & & \ddots & \ddots & & \vdots \\
 & & t_n & 2(t_n+t_{n-1}) & t_{n-1} & \\
 & & \cdots\cdots & & 0 & 1
\end{bmatrix}
\begin{bmatrix}
P_1' \\ P_2' \\ \\ \\ P_{n-1}' \\ P_n'
\end{bmatrix}
=
\begin{bmatrix}
P_1' \\
\frac{3}{t_2 t_3}\left(t_2^2(P_3-P_2)+t_3^2(P_2-P_1)\right) \\
\vdots \\
\frac{3}{t_{n-1}t_n}\left(t_{n-1}^2(P_n-P_{n-1})+t_n^2(P_{n-1}-P_{n-2})\right) \\
P_n'
\end{bmatrix}
$$

Curves

Parametric Curves

Cubic Splines

Solving for B_1 B_2 B_3 and B_4

$$B_{1k} = P_k,$$
$$B_{2k} = P_k^{'},$$
$$B_{3k} = \frac{3(P_{k+1} - P_k)}{t_{k+1}^2} - \frac{2P_k^{'}}{t_{k+1}} - \frac{P_{k+1}^{'}}{t_{k+1}}$$
$$B_{4k} = \frac{2(P_k - P_{k+1})}{t_{k+1}^3} + \frac{P_k^{'}}{t_{k+1}^2} + \frac{P_{k+1}^{'}}{t_{k+1}^2}$$

Curves

Parametric Curves

Cubic Splines

$$\begin{bmatrix} B_{1k} \\ B_{2k} \\ B_{3k} \\ B_{4k} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/t_{k+1}^3 & 1/t_{k+1}^2 \end{bmatrix} \begin{bmatrix} P_k \\ P_k' \\ P_{k+1} \\ P_{k+1}' \end{bmatrix}$$

$$P_k(t) = \sum_{i=1}^{4} B_{ik} t^{i-1} \qquad 0 \le t \le t_{k+1} \qquad 1 \le k \le n-1$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} B_{1k} & B_{2k} & B_{3k} & B_{4k} \end{bmatrix}^T$$

Curves

Parametric Curves

Cubic Splines

$$P_k(t)$$

$$= \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/t_{k+1}^2 & -2/t_{k+1} & 3/t_{k+1}^2 & -1/t_{k+1} \\ 2/t_{k+1}^3 & 1/t_{k+1}^2 & -2/t_{k+1}^3 & 1/t_{k+1}^2 \end{bmatrix} \begin{bmatrix} P_k \\ P_k' \\ P_{k+1} \\ P_{k+1}' \end{bmatrix}$$

$$= \big[(1 - 3t^2/t_{k+1}^2 + 2t^3/t_{k+1}^3) \quad (t - 2t^2/t_{k+1} + t^3/t_{k+1}^3)$$

$$(3t^2/t_{k+1}^2 - 2t^3/t_{k+1}^3) \quad (t^3/t_{k+1}^3 - t^2/t_{k+1}) \big] \begin{bmatrix} P_k & P_k' & P_{k+1} & P_{k+1}' \end{bmatrix}^T$$

Curves

Parametric Curves

Cubic Splines

Substituting $\boldsymbol{u=t/t_{k+1}}$ rearranging

$$P_k(u) = \begin{bmatrix} F_1(u) & F_2(u) & F_3(u) & F_4(u) \end{bmatrix} \begin{bmatrix} P_k & P_{k+1} & P_k' & P_{k+1}' \end{bmatrix}^T$$

$$0 \le u \le 1$$

$$1 \le k \le n-1$$

$$F_1(u) = 2u^3 - 3u^2 + 1$$

$$F_2(u) = -2u^3 + 3u^2$$

$$F_3(u) = u(u^2 - 2u + 1)t_{k+1}$$

$$F_4(u) = u(u^2 - u)t_{k+1}$$

F_1 ,F_2 ,F_3 ,F_4 are called the ***Blending* Functions**

Curves

Parametric Curves

Cubic Splines

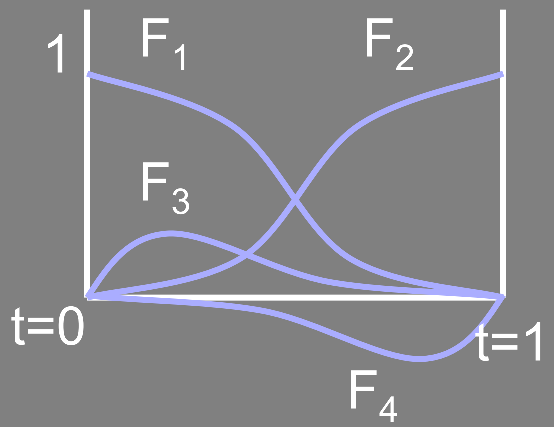
$$P_k(u)=[F][G]$$

Where F is the Blending function matrix and G Is the geometric information.

Curves

Parametric Curves

Cubic Splines



- $F_1(0)=1, F_2(0)=0, F_3(0)=0, F_4(0)=0$
  curve passes P_1
- $F_1(1)=0, F_2(1)=1, F_3(1)=0, F_4(1)=0$
  curve passes P_2
- $F_2=1-F_1, F_4=1-F_3$
- Relative magnitudes of $F_1, F_2 > F_3, F_4$

Curves

Parametric Curves

Cubic Splines

Piecewise Cubic Splines are determined by position vectors, tangent vectors and parameter value t_k .

The value of t_k can be chosen using either Chord Length parameterization or Uniform Parameterization.

If t_k=1 for all ***k*** then the Spline is called Normalized Spline.

Curves

Parametric Curves

Cubic Splines

Normalized Cubic Splines: $t_k=1$ for all segments
The blending functions thus become

$F_1(t) = 2t^3-3t^2+1$, $F_2(t) = -2t^3+3t^2$

$F_3(t) = t^3-2t^2+t$, $F_4(t) = t^3-t^2$

These are called *Hermite Polynomial Blending* functions

$$\begin{bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \\ F_4(t) \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Curves

Parametric Curves

Cubic Splines

The tridiagonal system for getting P' becomes

$$\begin{bmatrix} 1 & 0 & \cdots\cdots & & \\ 1 & 4 & 1 & & \\ \vdots & 1 & 4 & 1 & \vdots \\ \vdots & & \ddots & & \vdots \\ & & 1 & 4 & 1 \\ & & \cdots\cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} P_1' \\ P_2' \\ \vdots \\ \vdots \\ P_{n-1}' \\ P_n' \end{bmatrix} = \begin{bmatrix} P_1' \\ 3\left((P_3 - P_2) + (P_2 - P_1)\right) \\ \vdots \\ \\ 3\left((P_n - P_{n-1}) + (P_{n-1} - P_{n-2})\right) \\ P_n' \end{bmatrix}$$

Curves

Parametric Curves

Cubic Splines

End Conditions as:

- Clamped: P_1', P_n' known
- Relaxed/Natural: $P''(0) = 0$
  $P''(t_n)=0$
- Cyclic: $P_1'(0)=P_n'(t_n)$
  $P_1''(0)=P_n''(t_n)$
- Anticyclic: $P_1'(0) = -P_n'(t_n)$
  $P_1''(0) = -P_n''(t_n)$