Curves

B-Splines

Bezier Curve

There is no local control (change of one control point affects the whole curve)

Degree of curve is fixed by the number of control points

Curves

B-Splines

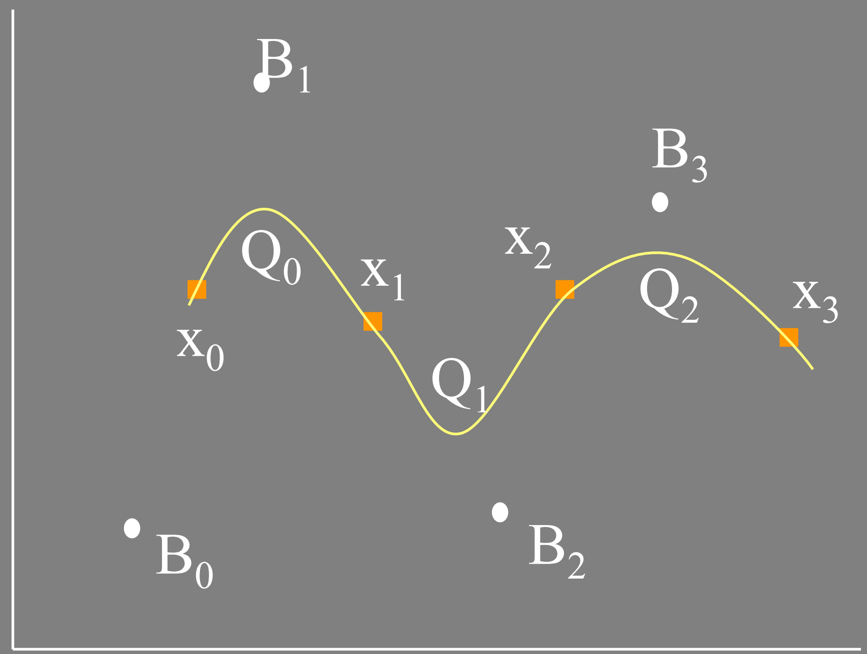
Each control point is associated with a unique basis function

Each point affects the shape of the curve over a range of parameter values where the basis function is non-zero

local control

Curves

B-Splines



Q_0:B_0 B_1 B_2 B_3
Q_1:B_1 B_2 B_3 B_4

Parameter t is defined as $x_i<t<x_{i+1}$

x_0 x_1 x_2 x_3: Knot values (knot vector)

- *Control Point*
- *Knot Point*

Curves

B-Splines

Polynomial spline function of order k (degree k-1)

$$P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t) \qquad t_{\min} \le t \le t_{\max}$$

$$2 \le k \le n+1$$

B_i : Control point

N_{ik} : Basis function

Curves

B-Splines

Cox-de Boor Recursive Formula

$$N_{i,1}(t) = \begin{cases} 1 & x_i \le t < x_{i+1} \\ 0 & otherwise \end{cases}$$

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

x_i' s are the knot values $x_i<x_{i+1}$

Curves

B-Splines

Properties

1. Partition of Unity

$$\sum_{i=1}^{n+1} N_{i,k}(t) = 1, N_{i,k} \geq 0$$

2. Covex hull property

Curves

B-Splines

Convex hull property

For a B-Spline curve of order *k* (degree *k-1*)
a point on the curve lies within the convex hull of
k neighboring points

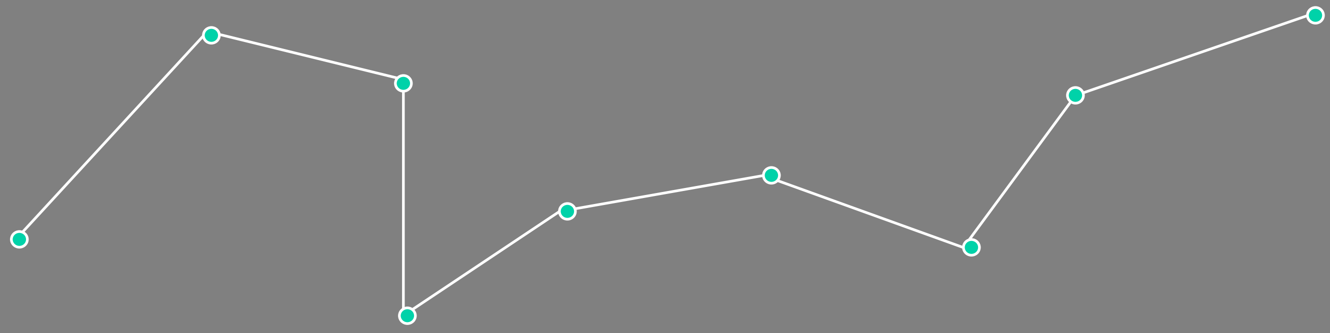
All points of B-Spline curve must lie within the
union of all such convex hulls

Curves

B-Splines

Convex hull property

K=2

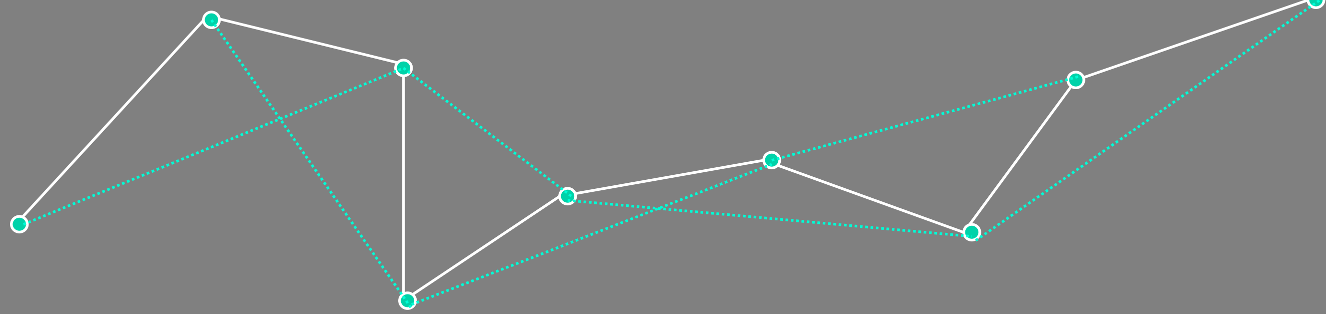


Curves

B-Splines

Convex hull property

K=3

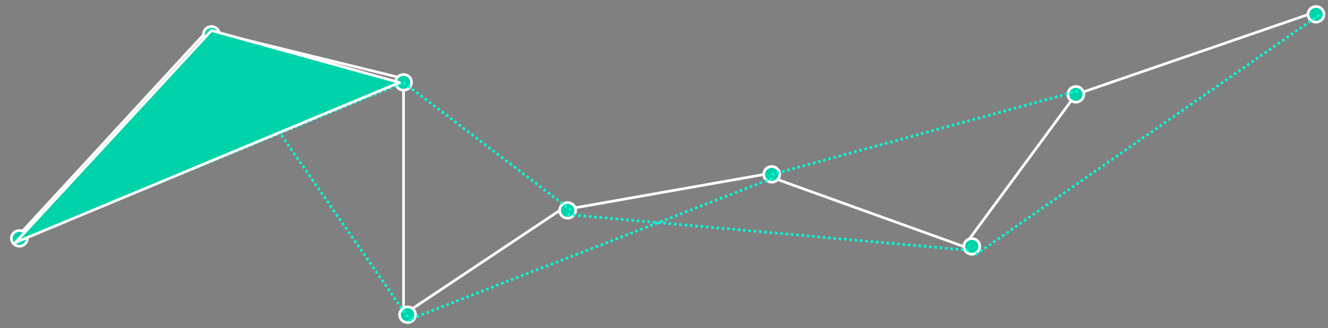


Curves

B-Splines

Convex hull property

K=3

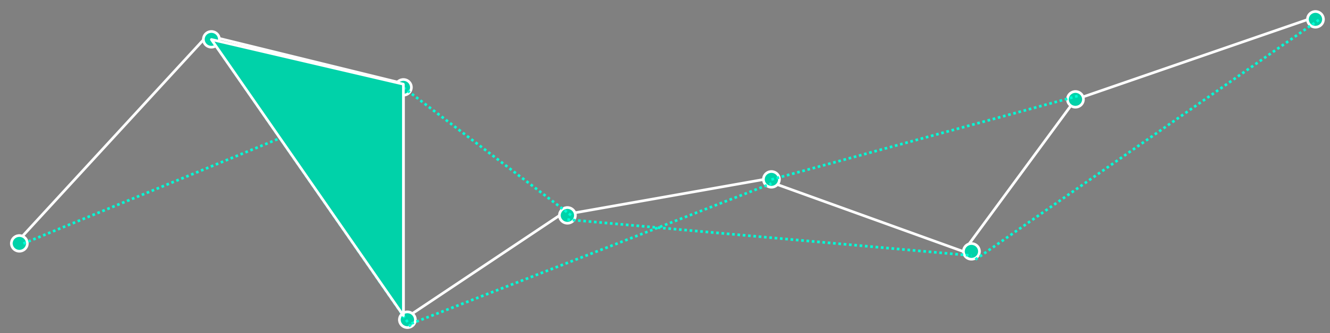


Curves

B-Splines

Convex hull property

K=3

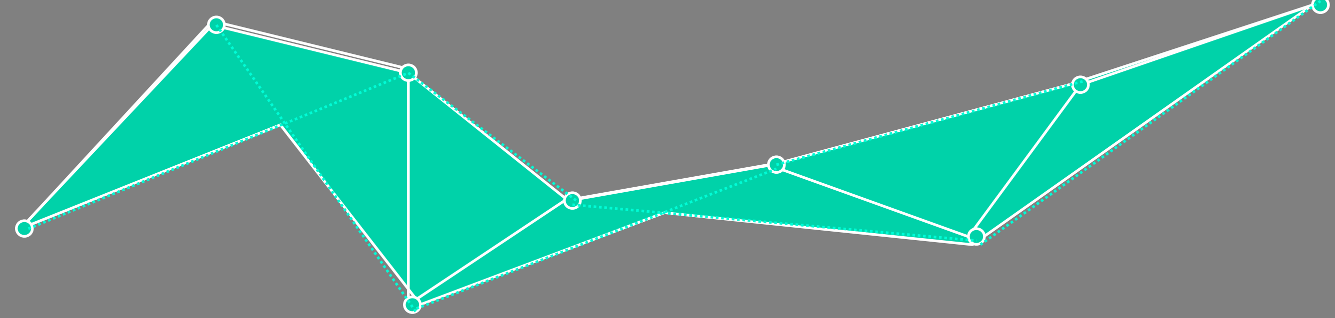


Curves

B-Splines

Convex hull property

K=3

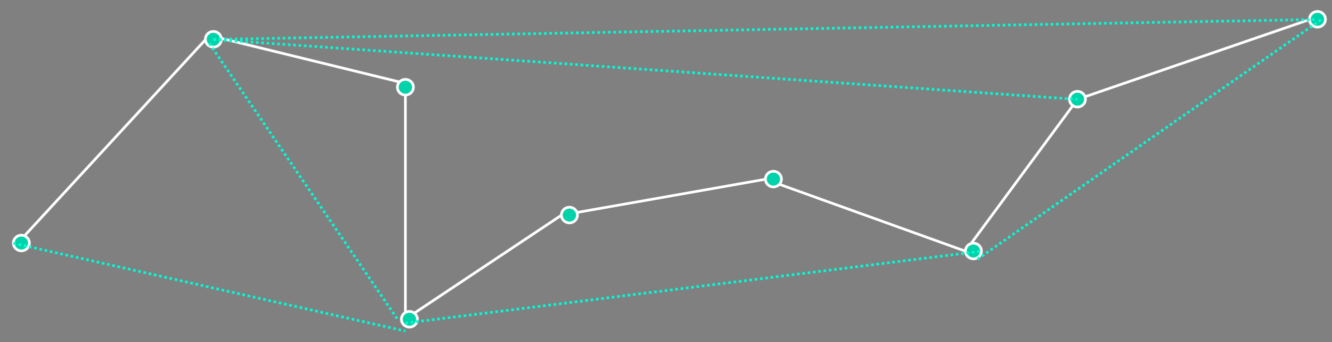


Curves

B-Splines

Convex hull property

K=8

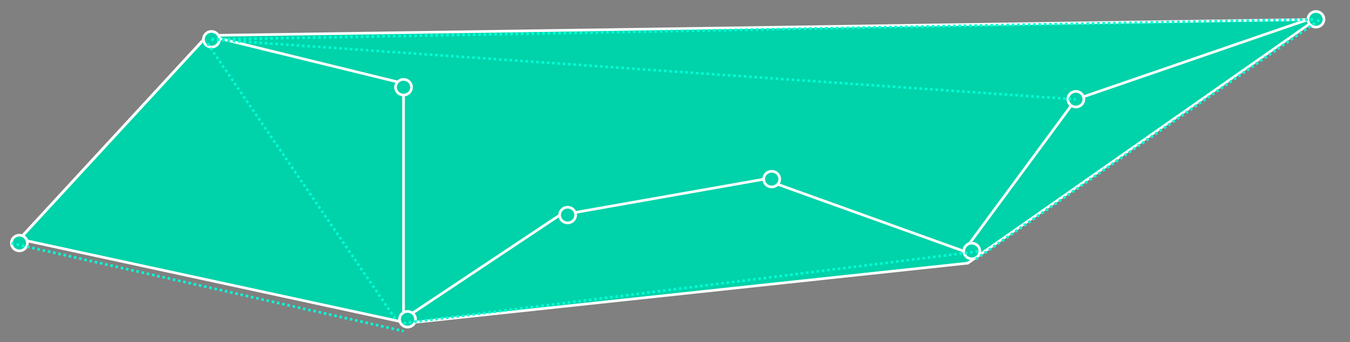


Curves

B-Splines

Convex hull property

K=8



Curves

B-Splines

Cox-de Boor Recursive Formula

$$N_{i,1}(t) = \begin{cases} 1 & x_i \le t < x_{i+1} \\ 0 & otherwise \end{cases}$$

$$N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}}$$

x_i' s are the knot values $x_i<x_{i+1}$

Curves

B-Splines

$Total\ knot\ values = n + 1 + k$

N_{ik} *has non zero value for* $x_i \rightarrow x_{i+k}$ *span*

Example

$n+1 = 4$, $k = 3$

$$N_{1,3} \quad \cdots \quad N_{4,3}$$

$$\downarrow \qquad\qquad \downarrow$$

$$x_1 - x_4 \qquad x_4 - x_7$$

knot vector $[x_1\ x_2\ \ldots\ x_7]$

Curves

B-Splines

Knot vector X can be:

- Uniform (periodic)
- Open-Uniform
- Non-Uniform

Curves

B-Splines

Uniform (Periodic)

Individual knot values are evenly spaced

e.g. [0 1 2 3 4]

[-0.2 -0.1 0 0.1 0.2]

[0 0.25 0.5 0.75 1]

Curves

B-Splines

Open

Has multiplicity of knot values at ends equal to the order k of the B-Spline basis function. Internal Knot values are evenly spaced

e.g. $k=2$ [0 0 1 2 3 4 4]
$k=3$ [0 0 0 1 2 3 3 3]
$k=4$ [0 0 0 0 1 2 2 2 2]

Curves

B-Splines

Non-Uniform

Unequal internal spacing and/or multiple internal knot(s)

e.g. [0 0.28 0.5 0.72 1]

[0 0 0 <u>1 1</u> 2 2 2]

[0 1 <u>2 2</u> 3 4]

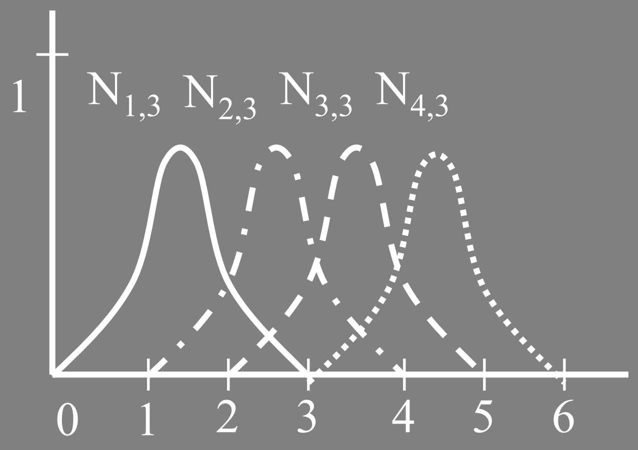
Curves

B-Splines

Uniform (Periodic)

Uniform Knot vectors yields periodic uniform basis functions

$N_{i,k}(t) = N_{i-1,k}(t-1) = N_{i+1,k}(t+1)$



$X = [\,0\;\;1\;\;2\;\;3\;\;4\;\;5\;\;6\,]$

$n + 1 = 4$

$k = 3$

Curves

B-Splines

Uniform (Periodic)

For k=3, n+1 = 4

Cox-de Boor Recursive Formula

$$N_{i,3}(t) = \frac{(t - x_i)N_{i,2}(t)}{x_{i+2} - x_i} + \frac{(x_{i+3} - t)N_{i+1,2}(t)}{x_{i+3} - x_{i+1}}$$

Curves

B-Splines

Uniform (Periodic)

$$N_{i,3}(t)$$

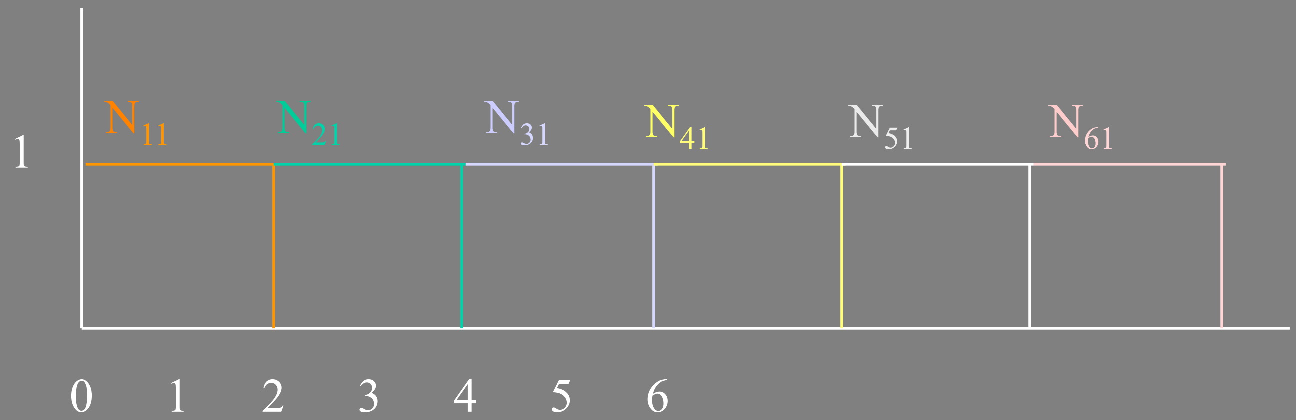
$$\frac{(t - x_i)N_{i,2}(t)}{x_{i+2} - x_i} \qquad \frac{(x_{i+3} - t)N_{i+1,2}(t)}{x_{i+3} - x_{i+1}}$$

$$\frac{(t - x_i)N_{i,1}(t)}{x_{i+1} - x_i} \qquad \frac{(x_{i+2} - t)N_{i+1,1}(t)}{x_{i+2} - x_{i+1}} \qquad \frac{(t - x_{i+1})N_{i+1,1}(t)}{x_{i+2} - x_{i+!}} \qquad \frac{(x_{i+3} - t)N_{i+2,1}(t)}{x_{i+3} - x_{i+2}}$$

Curves

B-Splines

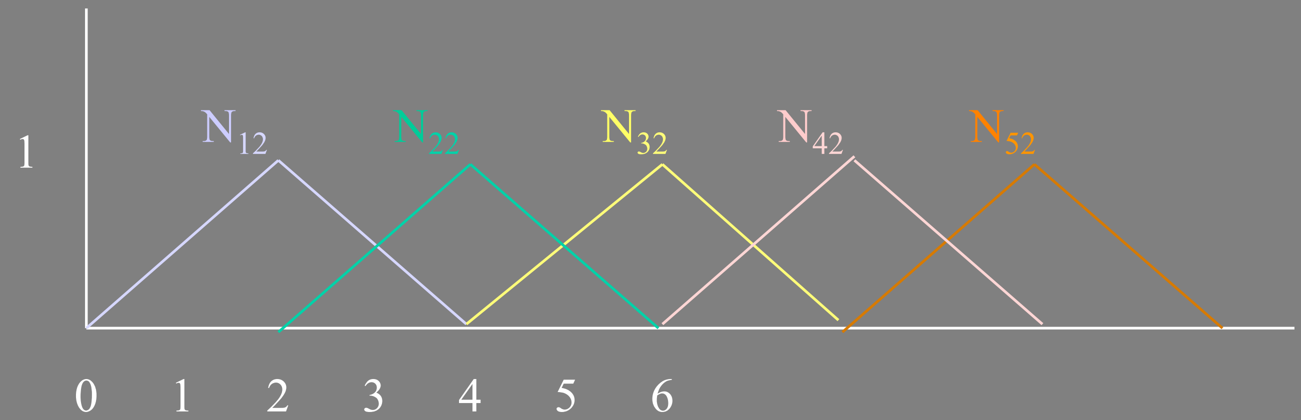
Uniform (Periodic)



Curves

B-Splines

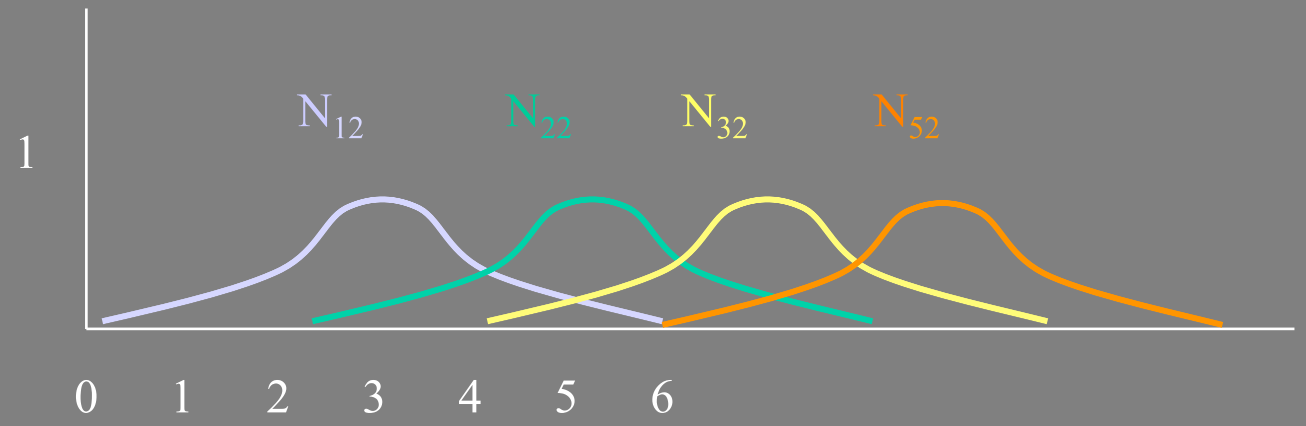
Uniform (Periodic)



Curves

B-Splines

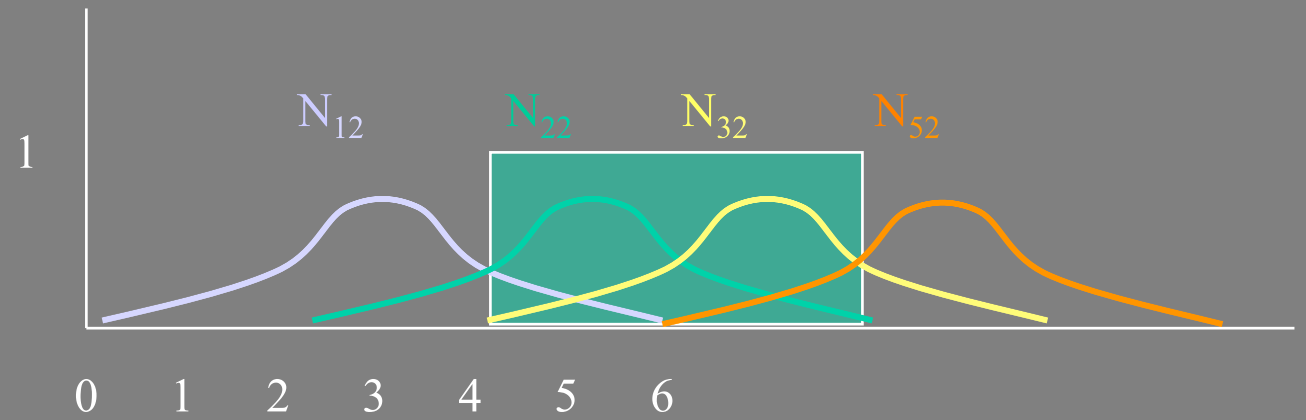
Uniform (Periodic)



Curves

B-Splines

Uniform (Periodic)



Curves

B-Splines

Open knot vector

$$x_i = 0 \qquad 1 \le x_i \le k$$

$$x_i = i - k \qquad k + 1 \le i \le n + 1$$

$$x_i = n - k + 2 \qquad n + 2 \le i \le n + k + 1$$

Curves

B-Splines

Open knot vector

When k=n+1 *Bezier Curve*
if n+1 = 4 = k knot vector = [0 0 0 0 1 1 1 1]

$$N_{1,4}(t) = (1-t)^3 = J_0^3$$

$$N_{2,4}(t) = 3t(1-t)^2 = J_1^3$$

$$N_{3,4}(t) = 3t^2(1-t) = J_2^3$$

$$N_{4,4}(t) = t^3 = J_3^3$$