Curves

B-Splines

Bezier Curve

There is no local control (change of one control point affects the whole curve)

Degree of curve is fixed by the number of control points
Curves

B-Splines

Each control point is associated with a unique basis function

Each point affects the shape of the curve over a range of parameter values where the basis function is non-zero

local control
Curves

B-Splines

Parameter $t$ is defined as $x_i < t < x_{i+1}$

$x_0$, $x_1$, $x_2$, $x_3$: Knot values (knot vector)

- **Control Point**
- **Knot Point**
Curves

B-Splines

Polynomial spline function of order $k$ (degree $k-1$)

$$P(t) = \sum_{i=1}^{n+1} B_i N_{i,k}(t)$$

$t_{\text{min}} \leq t \leq t_{\text{max}}$

$2 \leq k \leq n + 1$

$B_i$: Control point

$N_{i,k}$: Basis function
Curves

B-Splines

Cox-de Boor Recursive Formula

\[ N_{i,1}(t) = \begin{cases} 
1 & \text{if } x_i \leq t < x_{i+1} \\
0 & \text{otherwise} 
\end{cases} \]

\[ N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}} \]

The \( x_i \)'s are the knot values \( x_i < x_{i+1} \).
Curves

B-Splines

Properties
1. Partition of Unity
\[ \sum_{i=1}^{n+1} N_{i,k}(t) = 1, N_{i,k} \geq 0 \]

2. Covex hull property
Curves

B-Splines

Convex hull property

For a B-Spline curve of order $k$ (degree $k-1$), a point on the curve lies within the convex hull of $k$ neighboring points.

All points of B-Spline curve must lie within the union of all such convex hulls.
Curves

B-Splines

Convex hull property

K=2
Curves

B-Splines

Convex hull property

K=3
Curves

B-Splines

Convex hull property

K=3
Curves

B-Splines

Convex hull property

K=3
Curves

B-Splines
Convex hull property
K=3
Curves

B-Splines

Convex hull property

K=8
Curves

B-Splines

Convex hull property

K=8
Curves

B-Splines

Cox-de Boor Recursive Formula

\[ N_{i,1}(t) = \begin{cases} 
1 & \text{if } x_i \leq t < x_{i+1} \\
0 & \text{otherwise}
\end{cases} \]

\[ N_{i,k}(t) = \frac{(t - x_i)N_{i,k-1}(t)}{x_{i+k-1} - x_i} + \frac{(x_{i+k} - t)N_{i+1,k-1}(t)}{x_{i+k} - x_{i+1}} \]

\( x_i \)'s are the knot values \( x_i < x_{i+1} \)
Curves

B-Splines

Total knot values = \( n + 1 + k \)

\( N_{ik} \) has non zero value for \( x_i \rightarrow x_{i+k} \) span

Example

\( n+1 = 4, \ k = 3 \)

\[ \begin{align*}
N_{1,3} & \quad \cdots \quad N_{4,3} \\
\downarrow & \quad \downarrow \\
 x_1 - x_4 & \quad x_4 - x_7
\end{align*} \]

knot vector \([x_1 \ x_2 \ \cdots \ x_7]\)
Curves

B-Splines

Knot vector X can be:

- Uniform (periodic)
- Open-Uniform
- Non-Uniform
Curves

B-Splines

Uniform (Periodic)
Individual knot values are evenly spaced

e.g.  [ 0 1 2 3 4 ]
     [ -0.2 -0.1 0 0.1 0.2 ]
     [ 0 0.25 0.5 0.75 1 ]
Curves

B-Splines

Open

Has multiplicity of knot values at ends equal to the order $k$ of the B-Spline basis function. Internal Knot values are evenly spaced

\begin{align*}
\text{e.g.} & \quad k=2 & \begin{bmatrix} 0 & 0 & 1 & 2 & 3 & 4 & 4 \end{bmatrix} \\
& \quad k=3 & \begin{bmatrix} 0 & 0 & 0 & 1 & 2 & 3 & 3 & 3 \end{bmatrix} \\
& \quad k=4 & \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 2 & 2 & 2 \end{bmatrix}
\end{align*}
Curves

B-Splines

Non-Uniform
Unequal internal spacing and/or multiple internal knot(s)

e.g. \[ [0 \ 0.28 \ 0.5 \ 0.72 \ 1] \]
    \[ [0 \ 0 \ 0 \ 1 \ 1 \ 2 \ 2 \ 2] \]
    \[ [0 \ 1 \ 2 \ 2 \ 3 \ 4] \]
Curves

B-Splines

Uniform (Periodic)
Uniform Knot vectors yields periodic uniform basis functions
\[ N_{i,k}(t) = N_{i-1,k}(t-1) = N_{i+1,k}(t+1) \]

\[ X = [0 \ 1 \ 2 \ 3 \ 4 \ 5 \ 6] \]
\[ n + 1 = 4 \]
\[ k = 3 \]
Curves

B-Splines

Uniform (Periodic)

For \( k=3, \ n+1 = 4 \)

Cox-de Boor Recursive Formula

\[
N_{i,3}(t) = \frac{(t - x_i)N_{i,2}(t)}{x_{i+2} - x_i} + \frac{(x_{i+3} - t)N_{i+1,2}(t)}{x_{i+3} - x_{i+1}}
\]
Curves

B-Splines

Uniform (Periodic)

\begin{align*}
N_{i,3}(t) &= (t - x_i)N_{i,1}(t) \quad x_{i+2} - x_i \\
&+ (x_{i+3} - t)N_{i+1,2}(t) \quad x_{i+3} - x_{i+1} \\
&+ (x_{i+3} - t)N_{i+2,1}(t) \quad x_{i+3} - x_{i+2}
\end{align*}

\begin{align*}
N_{i,1}(t) &= (t - x_i)N_{i,1}(t) \quad x_{i+1} - x_i \\
&+ (x_{i+2} - t)N_{i+1,1}(t) \quad x_{i+2} - x_{i+1} \\
&+ (t - x_{i+1})N_{i+1,1}(t) \quad x_{i+2} - x_{i+1}
\end{align*}
Curves

B-Splines

Uniform (Periodic)
Curves

B-Splines

Uniform (Periodic)
Curves

B-Splines

Open knot vector

\[ x_i = 0 \quad 1 \leq x_i \leq k \]
\[ x_i = i - k \quad k + 1 \leq i \leq n + 1 \]
\[ x_i = n - k + 2 \quad n + 2 \leq i \leq n + k + 1 \]
Curves

B-Splines

Open knot vector

When $k = n + 1$ *Bezier Curve*
if $n + 1 = 4 = k$ knot vector $= [0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1]$

\[
\begin{align*}
N_{1,4}(t) &= (1 - t)^3 = J_0^3 \\
N_{2,4}(t) &= 3t(1 - t)^2 = J_1^3 \\
N_{3,4}(t) &= 3t^2(1 - t) = J_2^3 \\
N_{4,4}(t) &= t^3 = J_3^3
\end{align*}
\]