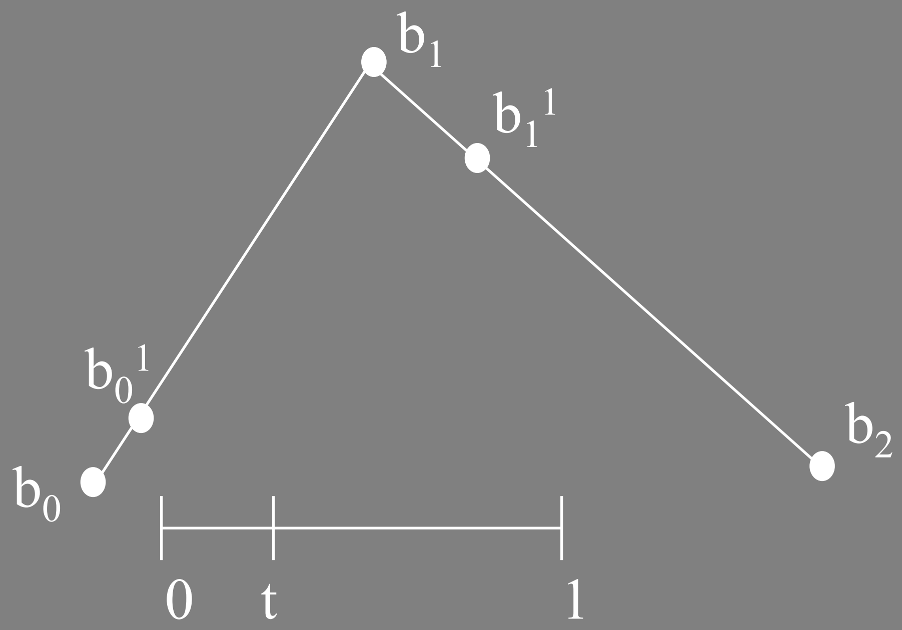
Curves

Bezier Curves

The de Casteljau ***Algorithm***

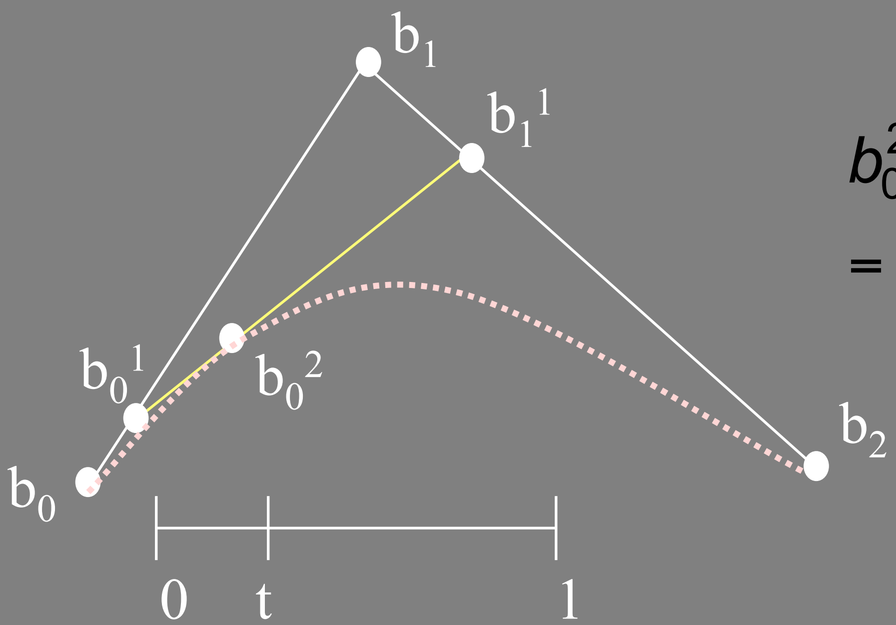


$$b_0^1(t) = (1-t)b_0 + tb_1$$
$$b_1^1(t) = (1-t)b_1 + tb_2$$

Curves

Bezier Curves

The **de Casteljau** *Algorithm*



$$b_0^2(t) = (1-t)b_0^1(t) + tb_1^1(t)$$
$$= (1-t)^2 b_0 + 2t(t-1)b_1 + t^2 b_2$$

Curves

Bezier Curves

The de Casteljau Algorithm

$$\text{Given}: b_0, b_1, \ldots, b_n$$

$$b_i^r(t) = (1-t)b_i^{r-1}(t) + t b_{i+1}^{r-1}(t)$$

$$r = 1, \ldots, n$$

$$i = 0, \ldots, n-r$$

$$b_i^0(t) = b_i$$

$$\begin{array}{llll}
b_0 & & & \\
b_1 & b_0^1 & & \\
b_2 & b_1^1 & b_0^2 & \\
b_3 & b_2^1 & b_1^2 & b_0^3
\end{array}$$

Curves

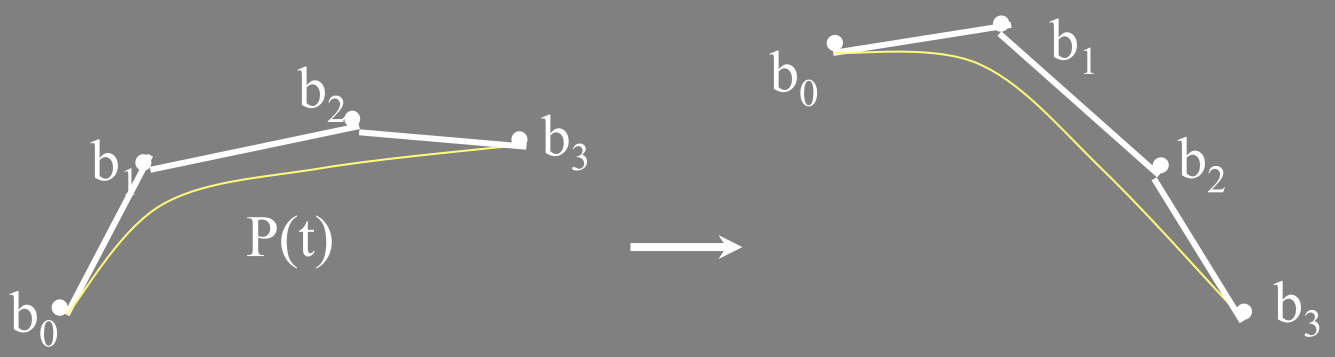
Bezier Curves

The de Casteljau Algorithm

Properties: Revisit

Affine invariance: Direct consequence of the algorithm

sequence of linear interpolation



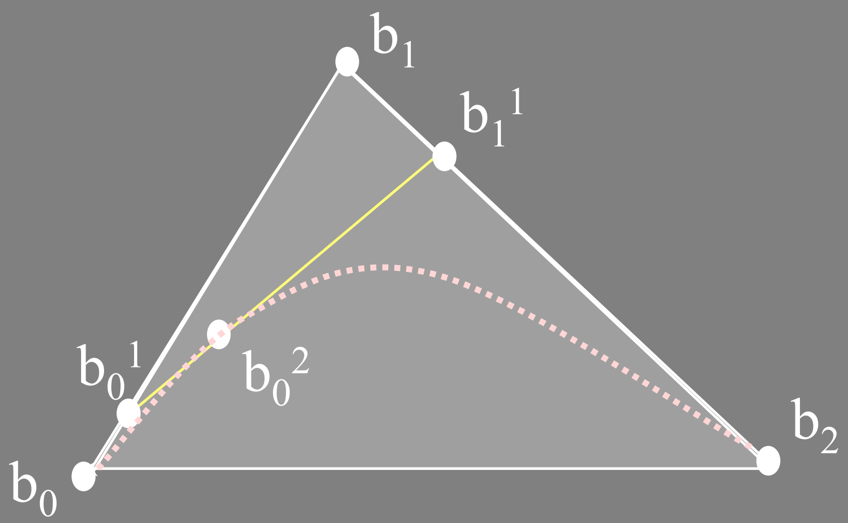
Curves

Bezier Curves

The de Casteljau Algorithm

Properties: Revisit

Convex hull



$P(t) = b_0^n(t) \quad t \in [0,1]$

Intermediate b_i^r *as convex combination of previous* b_i^{r-1}

Curves

Bezier Curves

The de Casteljau *Algorithm*

Properties: Revisit

Endpoint interpolation

$$P(0) = b_0^n(0) = b_0$$
$$P(1) = b_0^n(1) = b_n$$

Parameter transformation

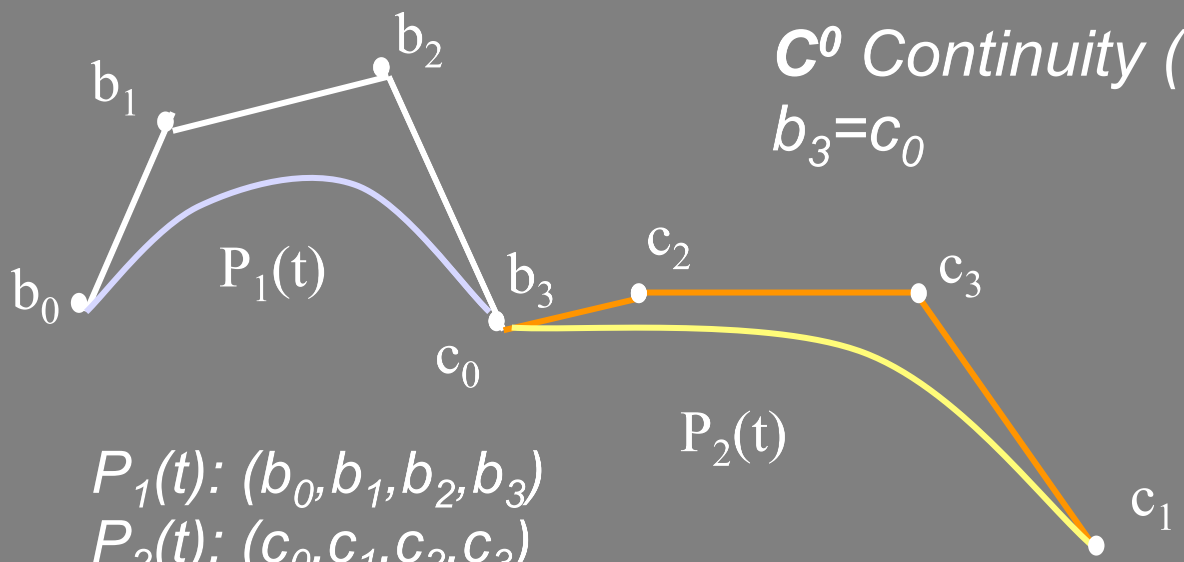
$$t \in [0,1]$$
$$u \in [a,b],\ t = \frac{u-a}{b-a}$$

Curves

Bezier Curves

Some More Properties

Continuity between adjacent Bezier Curves



C^0 Continuity (Positional)

$b_3 = c_0$

$P_1(t): (b_0, b_1, b_2, b_3)$

$P_2(t): (c_0, c_1, c_2, c_3)$

Curves

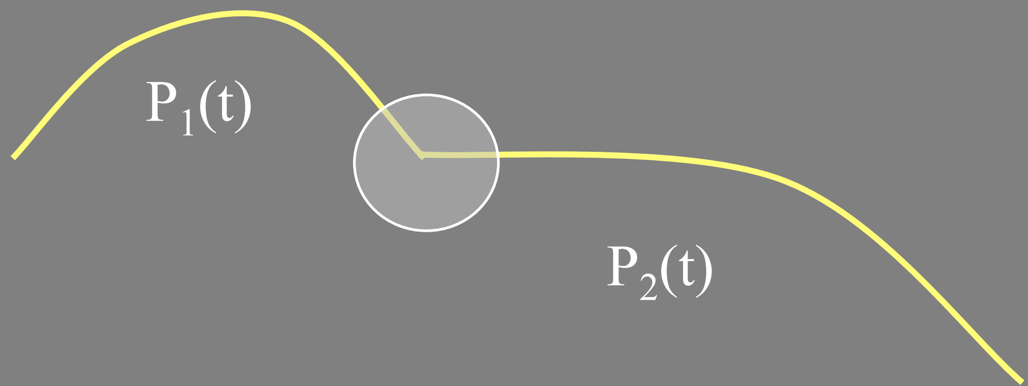
Bezier Curves

Some More Properties

Continuity between adjacent Bezier Curves

C^0 *Continuity (Positional)*

Not smooth

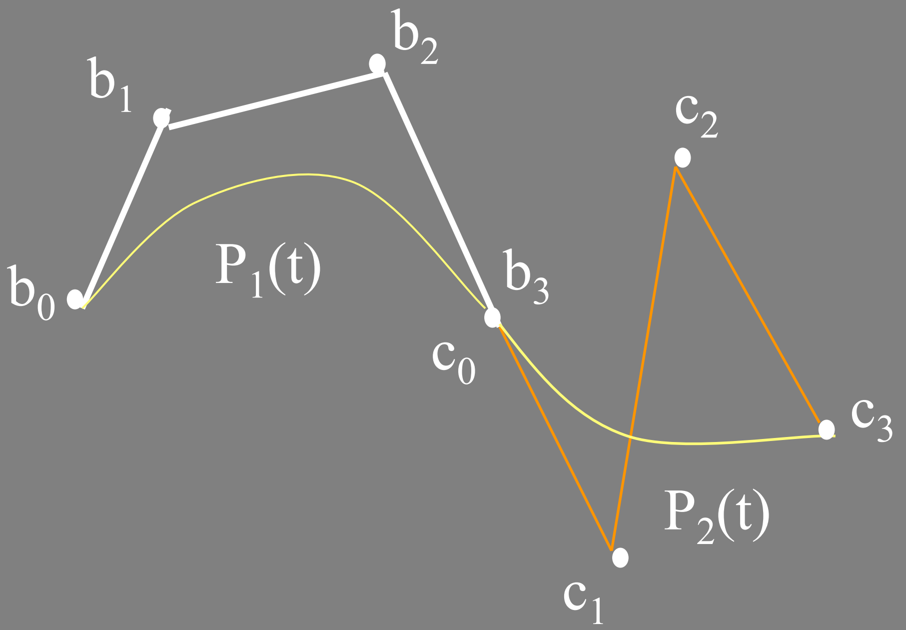


Curves

Bezier Curves

Some More Properties

Continuity between adjacent Bezier Curves



C^1 *Continuity (1st Derivative)*

$P_1'(1)=P_2'(0)$

For cubic

$3(b_3-b_2)=3(c_1-c_0)$

$c_1=b_3-b_2+c_0=2b_3-b_2$

Curves

Bezier Curves

Some More Properties

Continuity between adjacent Bezier Curves

C^2 *Continuity (2nd Derivative)*

$b_1 - 2b_2 + b_3 = c_0 - 2c_1 + c_2$

c_0, c_1 *are known from* ***C^0*** *and* ***C^1*** *continuity conditions*

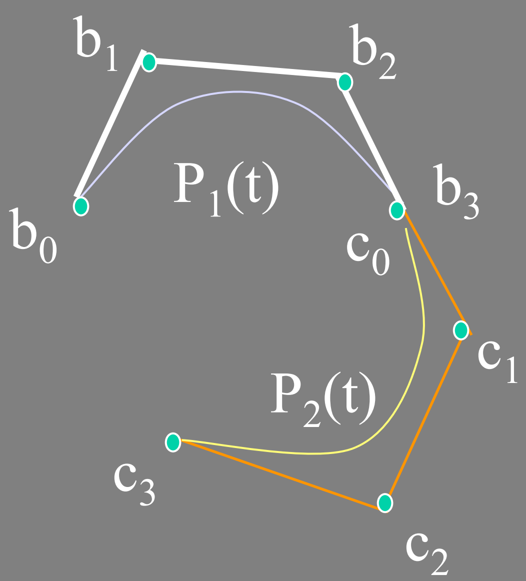
$c_2 = b_1 - 4(b_2 - b_3)$

Curves

Bezier Curves

Some More Properties

Continuity between adjacent Bezier Curves



C^2 Continuity (2nd Derivative)

$c_0 = b_3$

$c_1 - c_0 = b_3 - b_2$

$c_2 - b_1 = 4(b_3 - b_2)$

Curves

Bezier Curves

Degree Elevation

Increase degree of the curve

Increase the number of control points defining the polygon points

$$b_0, b_1, \ldots, b_n \Rightarrow b_0^*, b_1^*, \ldots, b_{n+1}^*$$

$$P(t) = \sum_{i=0}^{n} b_i J_i^n(t) = \sum_{i=0}^{n+1} b_i^* J_i^{n+1}(t)$$

Curves

Bezier Curves

Degree Elevation

$$P(t) = \sum_{i=0}^{n} b_i J_i^n(t) = \sum_{i=0}^{n+1} b_i^* J_i^{n+1}(t)$$

$$i.e., \sum_{i=0}^{n} b_i \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n+1} b_i^* \binom{n+1}{i} t^i (1-t)^{n+1-i}$$

Curves

Bezier Curves

Degree Elevation

$$P(t) = \sum_{i=0}^{n} b_i J_i^n(t) = \sum_{i=0}^{n+1} b_i^* J_i^{n+1}(t)$$

$$i.e., \sum_{i=0}^{n} b_i \binom{n}{i} t^i (1-t)^{n-i} = \sum_{i=0}^{n+1} b_i^* \binom{n+1}{i} t^i (1-t)^{n+1-i}$$

Multiplying LHS by (t+(1-t)) =1

$$\sum_{i=0}^{n} b_i \binom{n}{i} (t^i (1-t)^{n+1-i} + t^{i+1}(1-t)^{n-i}) = \sum_{i=0}^{n+1} b_i^* \binom{n+1}{i} t^i (1-t)^{n+1-i}$$

Curves

Bezier Curves

Degree Elevation

Comparing Coefficients

$$b_i^* \binom{n+1}{i} = b_i \binom{n}{i} + b_{i-1} \binom{n}{i-1}$$

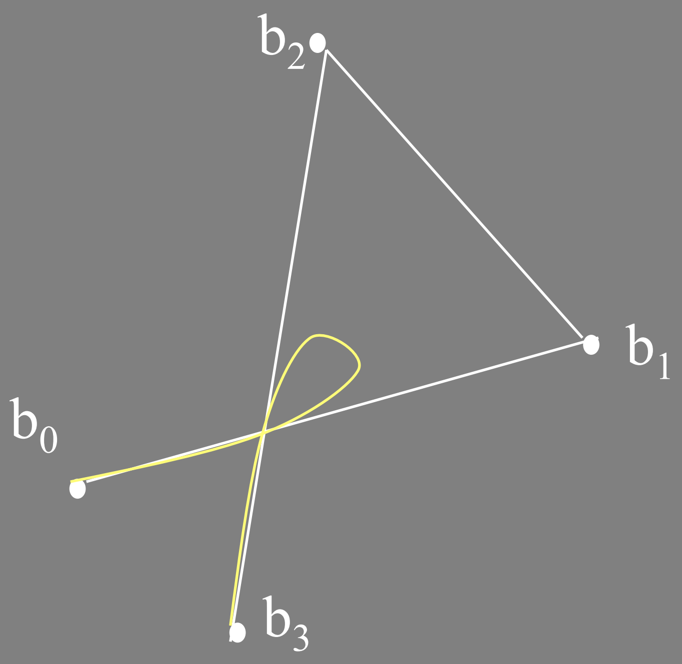
$$b_i^* = \frac{i}{n+1} b_{i-1} + \left(1 - \frac{i}{n+1}\right) b_i$$

$$\textit{Linear interpolation with } \alpha = \frac{i}{n+1}$$

Curves

Bezier Curves

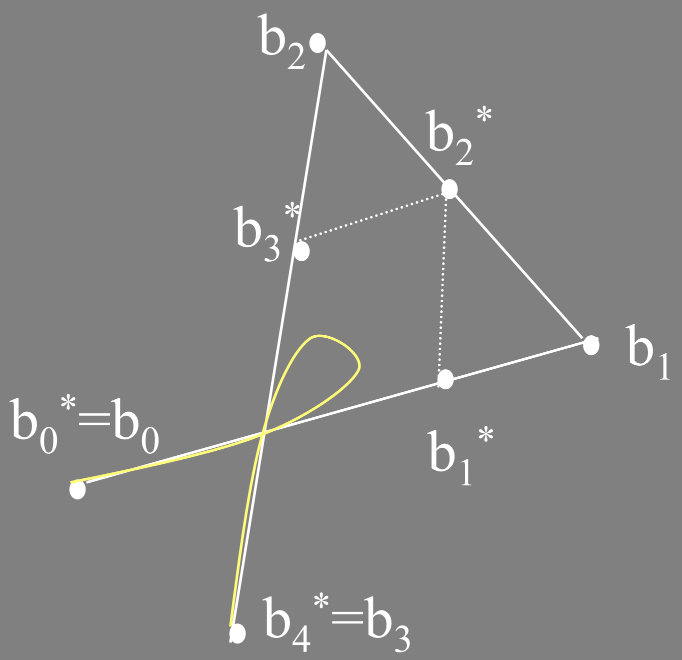
Degree Elevation



Curves

Bezier Curves

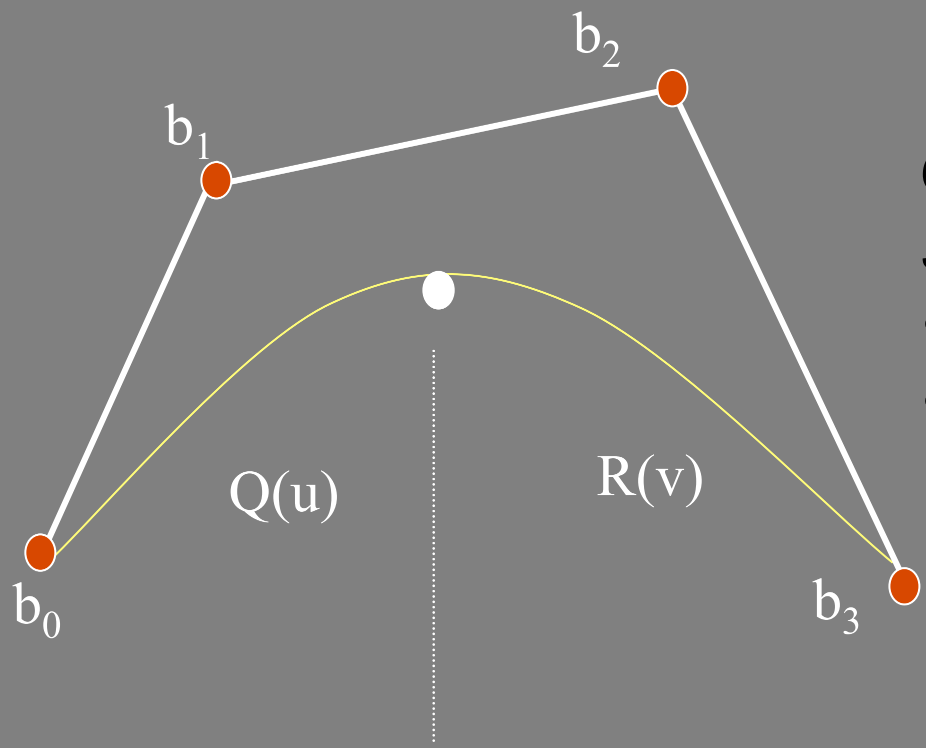
Degree Elevation



Curves

Bezier Curves

Subdivision



Curve P(t) 0 ≤ t ≤ 1 is subdivided into two curves

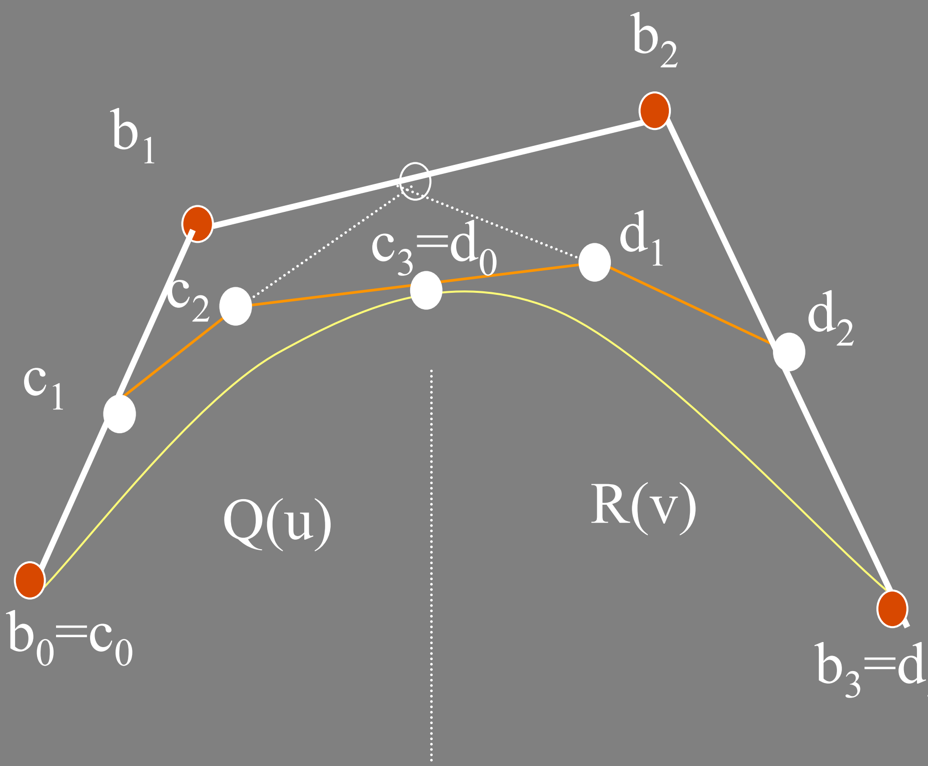
- *Q(u) 0 ≤ u ≤ 1*
- *R(v) 0 ≤ v ≤ 1*

Curves

Bezier Curves

Subdivision

Using de Casteljau Algorithm



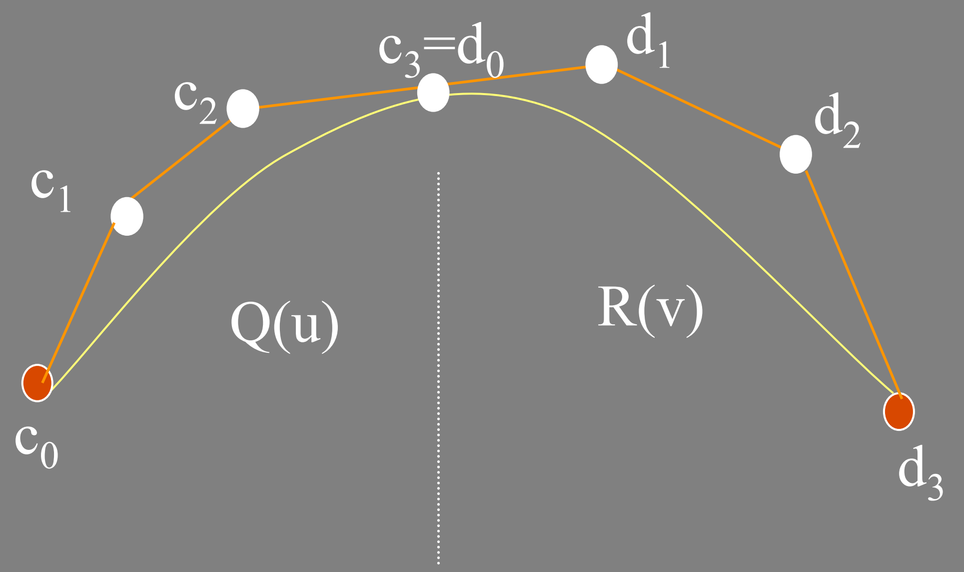
$c_0=b_0$
$c_1=(b_0+b_1)/2$
$c_2=(b_0+2b_1+b_2)/4$
$c_3=(b_0+3b_1+3b_2+b_3)/8$

$d_0=c_3$
$d_1=(b_1+2b_2+b_3)/4$
$d_2=(b_2+b_3)/2$
$d_3=b_3$

Curves

Bezier Curves

Subdivision



Curves

B-Splines

Bezier Curve

There is no local control (change of one control point affects the whole curve)

Degree of curve is fixed by the number of control points