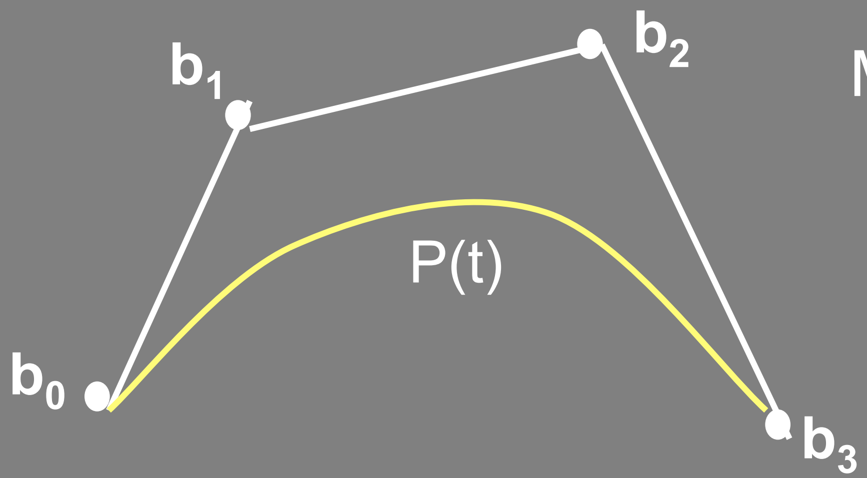
Parametric Curves

(Bezier Curves)

Curves

Bezier Curves

(Pierre Bezier -Renault Automobiles)



b_0 b_1 b_2 b_3 : Control Polygon

Mathematically

$$P(t) = \sum_{i=0}^{n} b_i J_i^n(t) \qquad 0 \le t \le 1$$

$J_{n,I}$ are called the Burnstein basis/blending functions

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Burnstein Polynomials

$$J_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}$$

$$\binom{n}{i} = \frac{n!}{i!(n-i)!}$$

$$J_{0,0}(t) = 1$$

$$J_i^n(t) = 0 \text{ for } i \notin \{0,\ldots,n\}$$

$$\sum_{i=0}^{n} J_i^n(t) = 1$$

$$J_i^n(t): \text{non-negative for } t \in [0,1]$$

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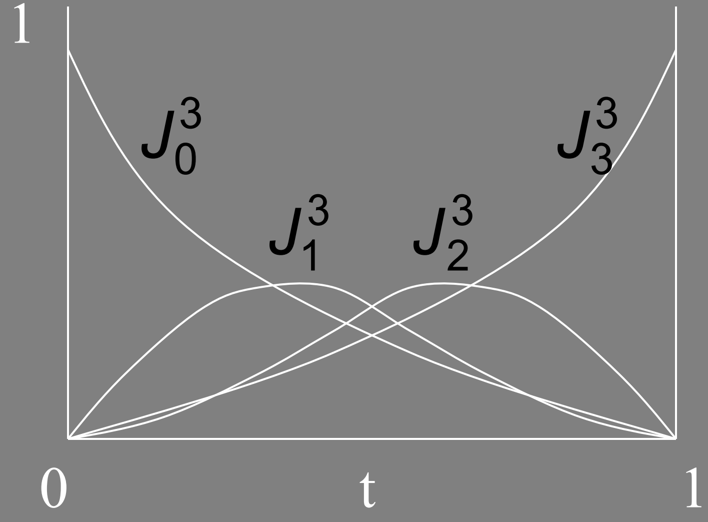
Cubic



$$J_0^3(t) = t^0(1-t)^3 = (1-t)^3$$
$$J_1^3(t) = 3t(1-t)^2$$
$$J_2^3(t) = 3t^2(1-t)$$
$$J_3^3(t) = t^3$$

$$P(t) = b_0 J_0^3 + b_1 J_1^3 + b_2 J_2^3 + b_3 J_3^3$$

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Cubic

$$P(t) = \begin{bmatrix} (1-t)^3 & 3t(1-t)^2 & 3t^2(1-t) & t^3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$$= \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

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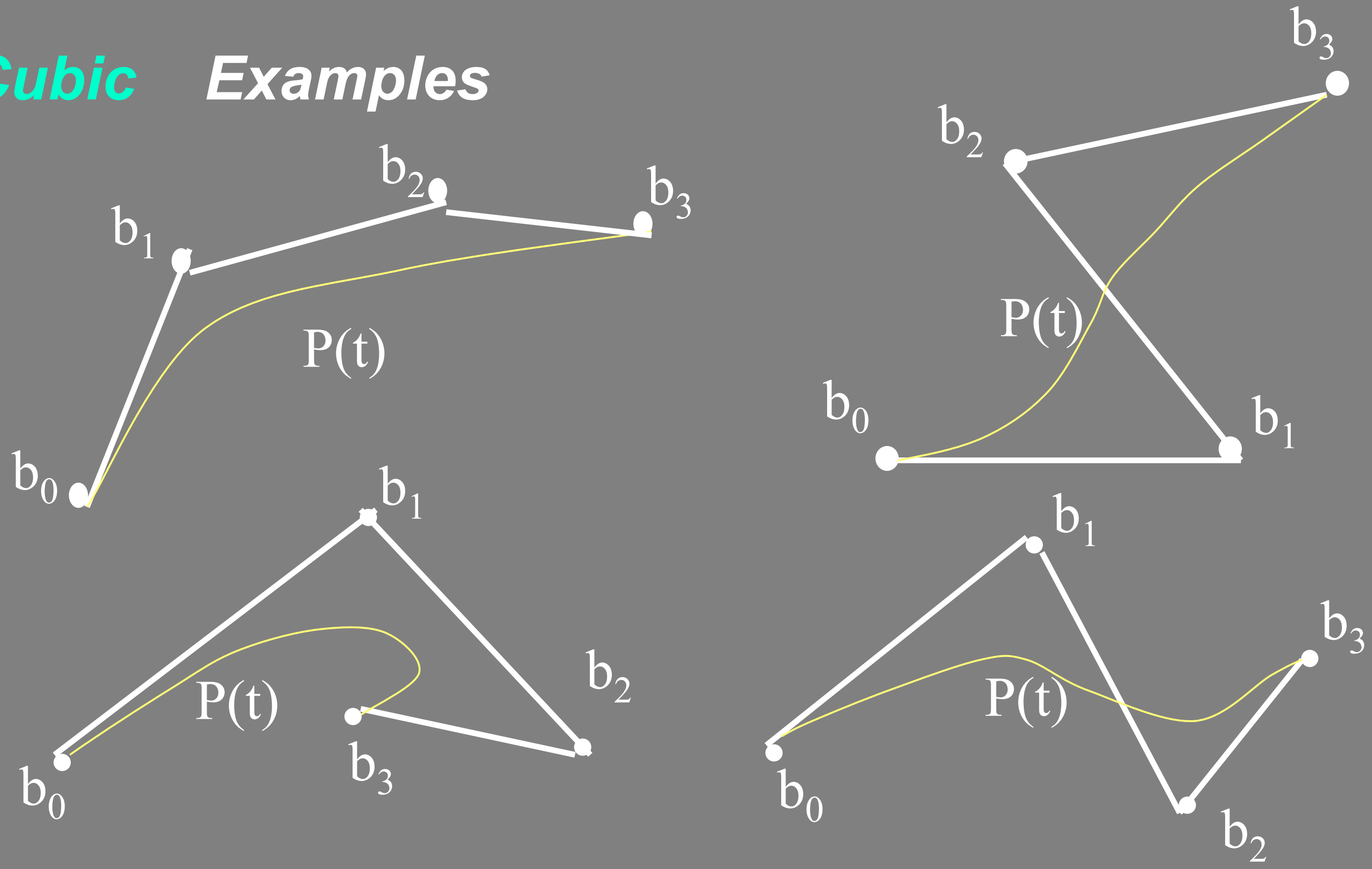
Cubic ***Examples***



Curves

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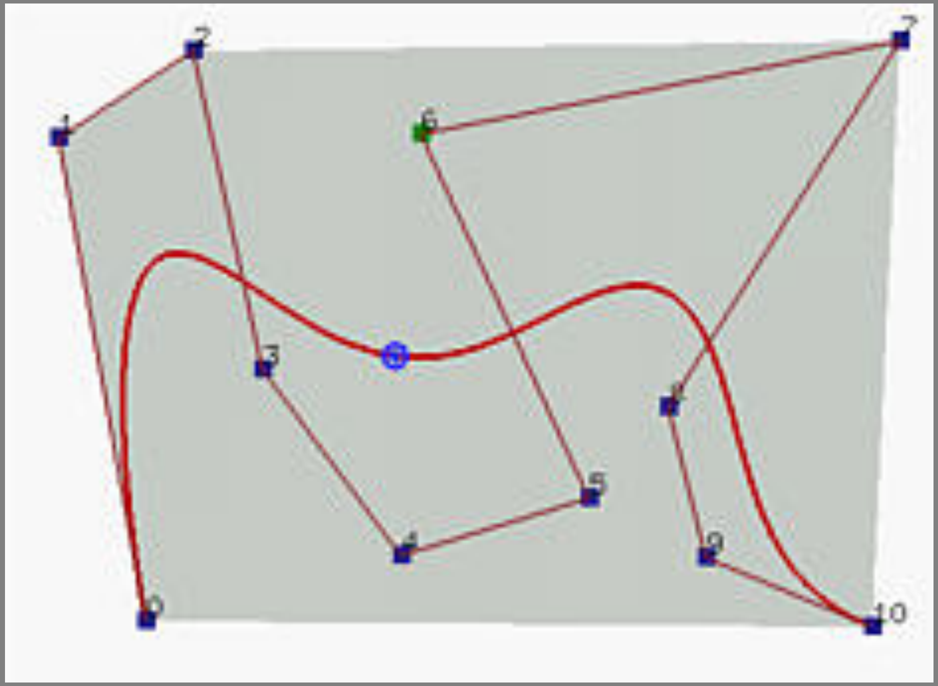
Cubic ***Examples***



Degree = 10

Curves

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Properties

End Point Interpolation

At t=0

$$i = 0, J_0^n(0) = \frac{n!}{n!(1)}(1)(1-0)^{n-0} = 1$$

$$i \neq 0, J_0^n(0) = \frac{n!}{i!(n-i)!}(0)^i(1-0)^{n-i} = 0$$

$$\Rightarrow P(0) = b_0 J_0^n(0) = b_0$$

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End Point Interpolation

At t=1

$$i = n, J_n^n(1) = \frac{n!}{n!(1)}(1)^n(0)^{n-n} = 1$$

$$i \neq n, J_i^n(1) = \frac{n!}{i!(n-i)!}(1)^i(1-1)^{n-i} = 0$$

$$\Rightarrow P(1) = b_n J_{n,n}(1) = b_n$$

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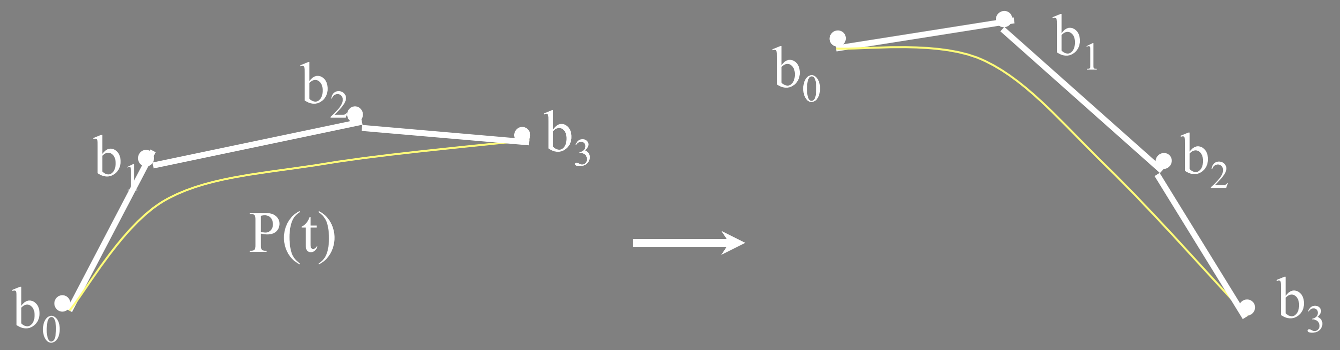
Properties

Affine Invariance

Applying an affine transformation to the curve is equivalent to applying the transformation to the control points.



Curves

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Properties

Affine Invariance

$$\varphi x = Ax + v$$

$$\varphi\left(\sum_i \alpha_i b_i\right) = A\left(\sum_i \alpha_i b_i\right) + v$$

$$= \sum_i \alpha_i A b_i + \sum_i \alpha_i v$$

$$= \sum_i \alpha_i (A b_i + v) = \sum_i \alpha_i \varphi b_i$$

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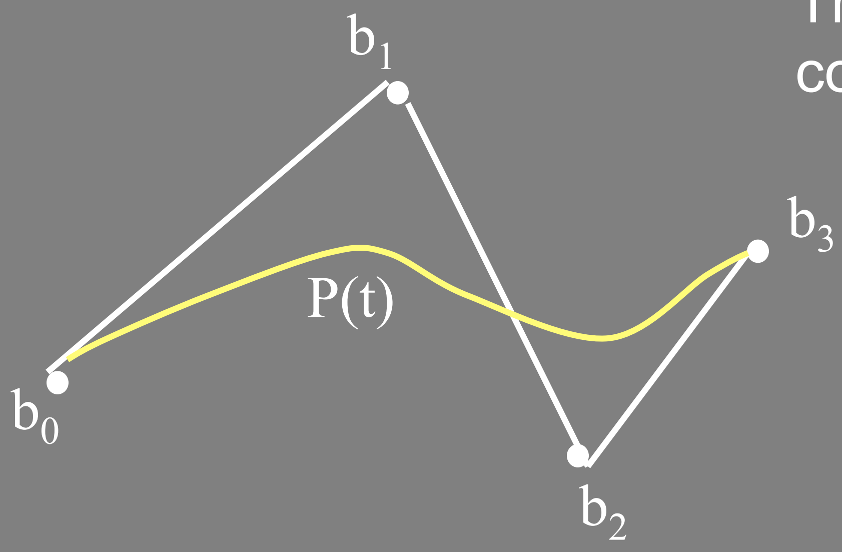
Properties

Convex hull



The curve lies in the
convex hull of the control points

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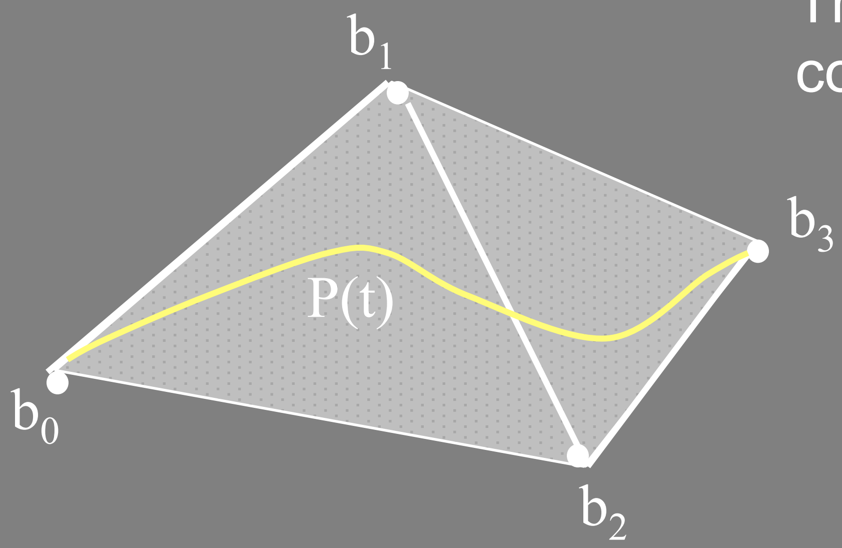
Convex hull



The curve lies in the
convex hull of the control points

$$\sum_{i=0}^{n} J_i^n(t) = 1$$

$$J_i^n(t): non-negative\ for$$
$$t \in [0,1]$$

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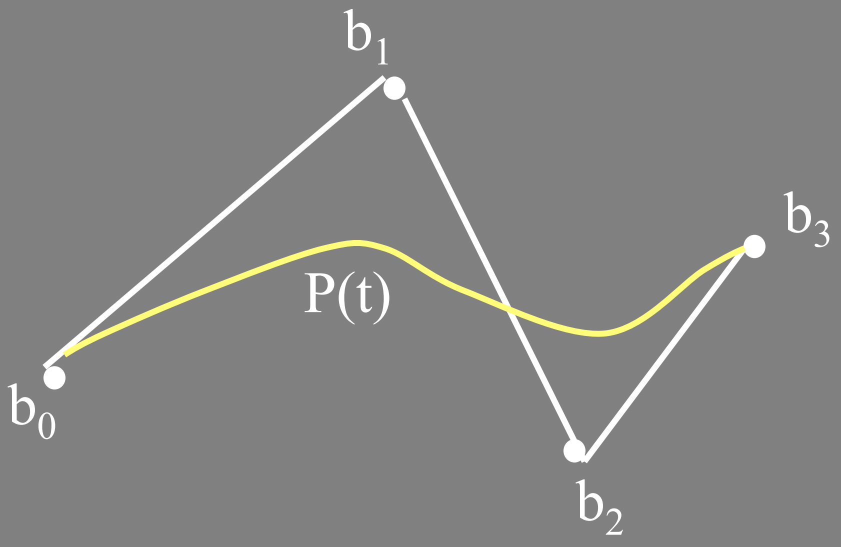
Properties

Symmetry



P(t) defined by $\mathbf{b_0, b_1, \ldots , b_n}$ is equal to P(1-t) defined by $\mathbf{b_n, b_{n-1}, \ldots , b_0}$

$$\sum_{i=0}^{n} b_i J_i^n(t) = \sum_{i=0}^{n} b_{n-i} J_i^n(1-t)$$

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Domain Parameter Transformation

$$t \in [0,1]$$
$$u \in [a,b]$$
$$t = \frac{u - a}{b - a}$$

$$\sum_{i=0}^{n} b_i J_i^n(t) = \sum_{i=0}^{n} b_i J_i^n\left(\frac{u-a}{b-a}\right)$$

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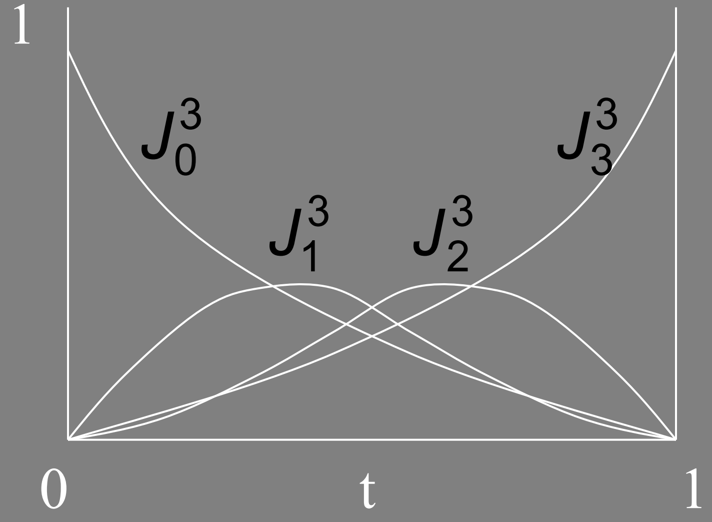
Pseudo-local Control



$J_i^n(t)$ has one maximum at

$$t = \frac{i}{n}$$

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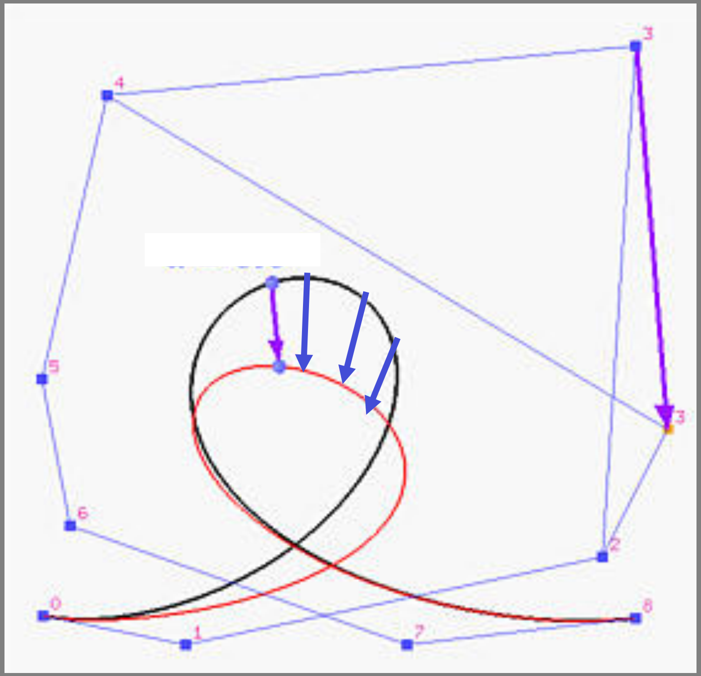
Pseudo-local Control



Curves

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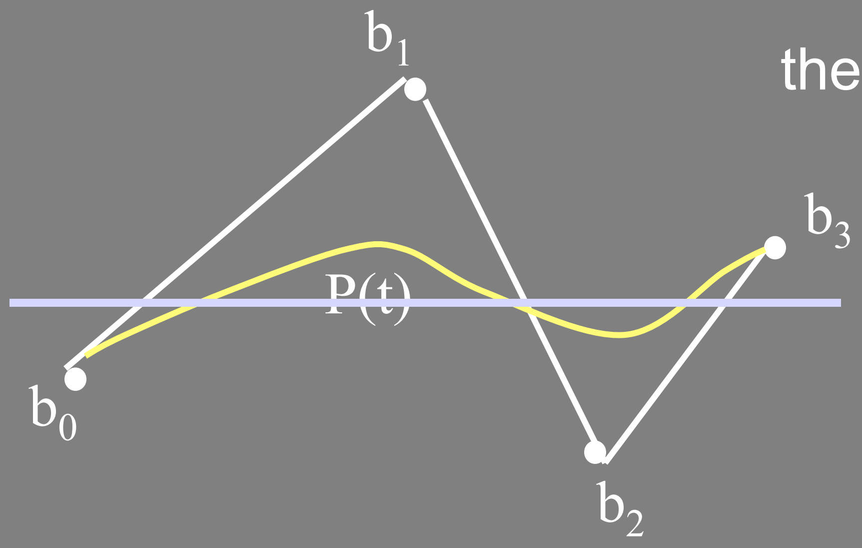
Properties

Variation Diminishing No straight line intersects a Bézier curve more times than it intersects the curve's control polyline



Curves

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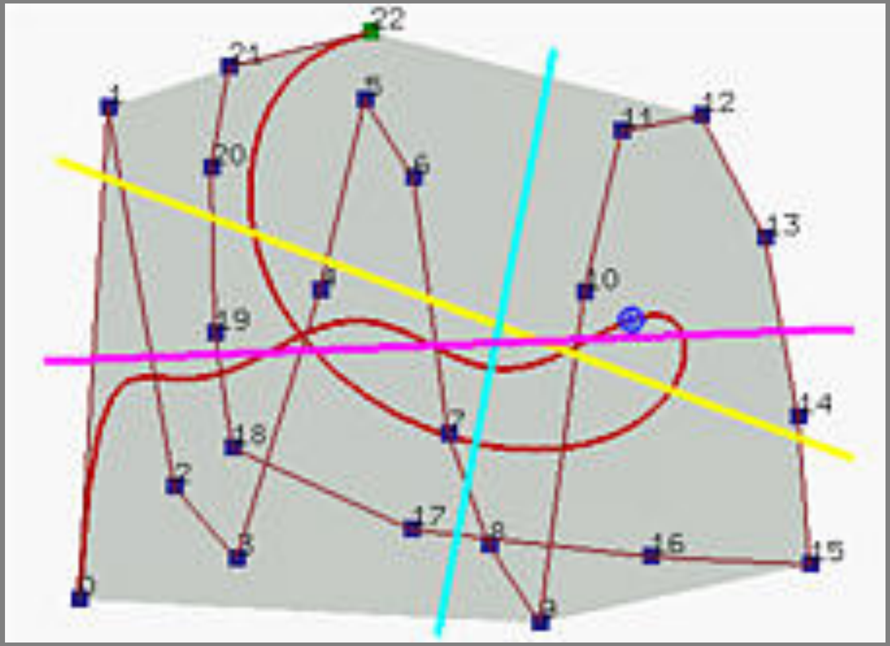
Properties

Variation Diminishing



Curves

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Properties

Tangent Vectors: Derivatives

$$\frac{d}{dt}P(t) = \sum_{i=0}^{n} \frac{d}{dt} J_i^n b_i \qquad \Rightarrow \frac{d}{dt} J_i^n(t) = \frac{d}{dt}\binom{n}{i} t^i (1-t)^{n-i}$$

$$\frac{d}{dt} J_i^n(t) = \binom{n}{i} i t^{i-1}(1-t)^{n-i} - \binom{n}{i}(n-i)t^i(1-t)^{n-i-1}$$

$$= n\binom{n-1}{i-1} t^{i-1}(1-t)^{n-i} - n\binom{n-1}{i} t^i (1-t)^{n-i-1}$$

$$= n(J_{i-1}^{n-1} - J_i^{n-1})$$

Curves

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Tangent Vectors: Derivatives

$$\frac{d}{dt}P(t) = n\sum_{i=0}^{n}(J_{i-1}^{n-1} - J_i^{n-1})b_i = n\sum_{i=1}^{n} J_{i-1}^{n-1}b_i - n\sum_{i=0}^{n-i} J_i^{n-1}b_i$$

$$= n\sum_{i=0}^{n-1} J_i^{n-1}b_{i+1} - n\sum_{i=0}^{n-i} J_i^{n-1}b_i = n\sum_{i=0}^{n-1}(b_{i+1} - b_i)J_i^{n-1}$$

Curves

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Tangent Vectors: Derivatives

$$P'(0) = n(b_1 - b_0)J_0^{n-1} = n(b_1 - b_0)$$

$$P'(1) = n(b_n - b_{n-1})J_{n-1}^{n-1} = n(b_n - b_{n-1})$$

Tangent Vectors at the ends of the curve have the same direction as the first and last polygon spans.

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Properties

Tangent Vectors: Derivatives