Beziers Curves
(Pierre Bezier - Renault Automobiles)

Mathematically

\[ P(t) = \sum_{i=0}^{n} b_i J_{i}^n(t) \quad 0 \leq t \leq 1 \]

\( J_{n,i} \) are called the Bernstein basis/blending functions
Bezier Curves

**Burnstein Polynomials**

\[
J_i^n(t) = \binom{n}{i} t^i (1-t)^{n-i}
\]

\[
\binom{n}{i} = \frac{n!}{i!(n-i)!}
\]

\[
J_{0,0}(t) = 1
\]

\[
J_i^n(t) = 0 \text{ for } i \not\in \{0,\ldots,n\}
\]

\[
\sum_{i=0}^{n} J_i^n(t) = 1
\]

\[
J_i^n(t) : \text{non-negative for } t \in [0,1]
\]
Bezier Curves

Cubic

\[ P(t) = b_0 J_0^3 + b_1 J_1^3 + b_2 J_2^3 + b_3 J_3^3 \]

\[ J_0^3(t) = t^0 (1 - t)^3 = (1 - t)^3 \]
\[ J_1^3(t) = 3t(1 - t)^2 \]
\[ J_2^3(t) = 3t^2 (1 - t) \]
\[ J_3^3(t) = t^3 \]
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**Cubic**

\[ P(t) = \begin{bmatrix} (1 - t)^3 & 3t(1 - t)^2 & 3t^2(1 - t) & t^3 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} t^3 & t^2 & t & 1 \end{bmatrix} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix} \]
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**Cubic Examples**

\[
P(t) = b_0 + b_1 t + b_2 t^2 + b_3 t^3
\]
Curves

Bezier Curves

Examples

Degree = 10
Bezier Curves

**Properties**

**End Point Interpolation**

At $t=0$

\[ i = 0, J_0^n(0) = \frac{n!}{n!(1)} (1)(1 - 0)^{n-0} = 1 \]

\[ i \neq 0, J_0^n(0) = \frac{n!}{i!(n - i)!} (0)^i (1 - 0)^{n-i} = 0 \]

\[ \Rightarrow P(0) = b_0 J_0^n(0) = b_0 \]
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Properties

End Point Interpolation

At $t=1$

$$i = n, J_n^n(1) = \frac{n!}{n!(1)} (1)^n (0)^{n-n} = 1$$

$$i \neq n, J_i^n(1) = \frac{n!}{i!(n-i)!} (1)^i (1-1)^{n-i} = 0$$

$$\Rightarrow P(1) = b_n J_{n,n}(1) = b_n$$
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**Properties**

**Affine Invariance**

Applying an affine transformation to the curve is equivalent to applying the transformation to the control points.
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Properties

Affine Invariance

\[ \varphi x = Ax + v \]

\[ \varphi \left( \sum_i \alpha_i b_i \right) = A \left( \sum_i \alpha_i b_i \right) + v \]

\[ = \sum_i \alpha_i Ab_i + \sum_i \alpha_i v \]

\[ = \sum_i \alpha_i (Ab_i + v) = \sum_i \alpha_i \varphi b_i \]
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**Properties**

**Convex hull**

The curve lies in the convex hull of the control points.
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**Properties**

**Convex hull**

The curve lies in the convex hull of the control points

\[ P(t) \]

\[ b_0, b_1, b_2, b_3 \]

\[ \sum_{i=0}^{n} J_i^n(t) = 1 \]

\[ J_i^n(t) : \text{non-negative for} \quad t \in [0,1] \]
Bezier Curves

Properties

Symmetry

$P(t)$ defined by $b_0, b_1, \ldots, b_n$ is equal to $P(1-t)$ defined by $b_n, b_{n-1}, \ldots, b_0$

$$
\sum_{i=0}^{n} b_i \mathcal{J}_i^n(t) = \sum_{i=0}^{n} b_{n-i} \mathcal{J}_i^n(1-t)
$$
Bezier Curves

**Properties**

**Domain Parameter Transformation**

\[
t \in [0, 1] \\
u \in [a, b] \\
t = \frac{u - a}{b - a}
\]

\[
\sum_{i=0}^{n} b_i J_i^n(t) = \sum_{i=0}^{n} b_i J_i^n\left(\frac{u - a}{b - a}\right)
\]
Bezier Curves

Properties

Pseudo-local Control

\[ J_i^n (t) \] has one maximum at

\[ t = \frac{i}{n} \]
Curves

Bezier Curves

Properties

Pseudo-local Control
Bezier Curves

Properties

Variation Diminishing

No straight line intersects a Bézier curve more times than it intersects the curve's control polyline.
Curves

Bezier Curves

*Properties*

Variation Diminishing
Beziers Curves

Properties

Tangent Vectors: Derivatives

\[
\frac{d}{dt} P(t) = \sum_{i=0}^{n} \frac{d}{dt} J_i^n b_i \quad \Rightarrow \quad \frac{d}{dt} J_i^n(t) = \frac{d}{dt} \binom{n}{i} t^i (1-t)^{n-i}
\]

\[
\frac{d}{dt} J_i^n(t) = \binom{n}{i} it^{i-1}(1-t)^{n-i} - \binom{n}{i} (n-i)t^i (1-t)^{n-i-1}
\]

\[
= n \binom{n-1}{i-1} t^{i-1}(1-t)^{n-i} - n \binom{n-1}{i} t^i (1-t)^{n-i-1}
\]

\[
= n(J_{i-1}^n - J_i^n)
\]
Beziers Curves

Properties

Tangent Vectors: Derivatives

\[
\frac{d}{dt} P(t) = n \sum_{i=0}^{n} (J_{i-1}^{n-1} - J_{i}^{n-1}) b_i = n \sum_{i=1}^{n} J_{i-1}^{n-1} b_i - n \sum_{i=0}^{n-i} J_{i}^{n-1} b_i \\
= n \sum_{i=0}^{n-1} J_{i}^{n-1} b_{i+1} - n \sum_{i=0}^{n-i} J_{i}^{n-1} b_i = n \sum_{i=0}^{n-1} (b_{i+1} - b_i) J_{i}^{n-1}
\]
Beziers Curves

Properties

Tangent Vectors: Derivatives

\[ P'(0) = n(b_1 - b_0)J_0^{n-1} = n(b_1 - b_0) \]
\[ P'(1) = n(b_n - b_{n-1})J_{n-1}^{n-1} = n(b_n - b_{n-1}) \]

Tangent Vectors at the ends of the curve have the same direction as the first and last polygon spans.
Bezier Curves

Properties

Tangent Vectors: Derivatives

\[ \Delta b_0 \quad \Delta b_1 \quad \Delta b_2 \]

Origin