LOCAL SEARCH HEURISTICS FOR FACILITY LOCATION PROBLEMS

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Local Search Heuristics for Facility Location Problems

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Certificate

This is to certify that the thesis titled "Local Search Heuristics for Facility Location Problems" being submitted by Vinayaka Pandit to the Indian Institute of Technology Delhi, for the award of the degree of Doctor of Philosophy, is a record of bona-fide research work carried out by him under my supervision. In my opinion, the thesis has reached the standards fulfilling the requirements of the regulations relating to the degree.

The results contained in this thesis have not been submitted in part or full to any other university or institute for the award of any degree/diploma.

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To my parents and my sister
for their love and constant support

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Abstract

In this thesis, we develop approximation algorithms for facility location problems based on local search techniques. Facility location is an important problem in operations research. Heuristic approaches have been used to solve many variants of the problem since the 1960s. The study of approximation algorithms for facility location problems started with the work of Hochbaum. Although local search is a popular heuristic among practitioners, their analysis from the point of view of approximation started only recently. In a short time, local search has emerged as a versatile technique for obtaining approximation algorithms for facility location problems. Significantly, there are many variants for which local search is the only technique known to give constant factor approximations. In this thesis, we demonstrate the effectiveness of local search for facility location by obtaining approximation algorithms for many diverse variants of the problem.

Local search is an iterative heuristic used to solve many optimization problems. Typically, a local search heuristic starts with any feasible solution, and improves the quality of the solution iteratively. At each step, it considers only local operations to improve the cost of the solution. A solution is called a *local minima* if there is no local operations which improves the cost. One of the earliest and most popular local search heuristic for facility location was proposed by Kuehn and Hamburger in the 1960s. However, the analysis of a local minima for the worst case ratio of its cost to the cost of the optimal solution began only recently with the work of Korupolu, Plaxton, and Rajaraman. Since then, their analysis has been improved, and local search heuristics for diverse variants of facility location have been presented. Informally, *locality gap* is the worst case ratio of the cost of a local minima to the cost of the global optima.

We present the first analysis of a local search algorithm which gives a constant factor approximation for the k-median problem while opening at most k facilities. Our analysis yields a $3(1+\epsilon)$ approximation algorithm for the k-median problems which is the best known ratio currently. We show that our technique can be used to analyze local search algorithms for the uncapacitated facility location problem, capacitated facility location problem with soft capacities, k-uncapacitated facility location problem, and a bi-criteria facility location problem. Our analysis yields $3(1+\epsilon)$ approximation algorithm for the uncapacitated facility location, $4(1+\epsilon)$ approximation for the capacitated facility location with soft capacities, and $5(1+\epsilon)$ approximation for the k-uncapacitated facility location problem. We establish an interesting connection between the price of anarchy of a service provider game and the locality gap of k-uncapacitated facility location problem. This gives rise to the possibility of reducing the question of upper bounding the price of anarchy of certain games to

the question of upper bounding the locality gap of their corresponding optimization problems.

Contents

List of Figures

Chapter 1

Introduction

Optimizing the cost of repetitive tasks is an important exercise in operations research. Organizations of even modest sizes optimize their operations to minimize costs and improve efficiency. One of the many aspects of operations is cost effective and efficient accessing of a set of services or infrastructural facilities by a group of demand points or clients. Typically, many service locations are set up, each of which serves the demands of a subset of demand points. Some examples of such an exercise are setting up of a supply chain of a business, locating essential services such as health care and education, and construction of transportation networks. Facility location problems proposed in operations research provide mathematical formulations of the common optimization aspects of these problems. We begin by describing example scenarios where facility location formulations are natural.

Consider an automobile service company which aims to provide its customer with efficient access to service stations. To do so, it would like to ensure that, all its customers have a nearby service station. However, the company incurs significant cost in setting up each service station. So, opening a large number of service station may be prohibitively costly. Ideally, the company would like to open a fixed number of service stations such that the average distance of its customers to their nearest service station is minimum.

Consider the problem of organizing training camps for a large group of people. Most of the costs involved including setting up of the camps, and the logistics of transport for the people are incurred just once. In such a scenario, one would like to open a set of training facilities and assign people to the training facilities such that the cost of the entire exercise, i.e, the cost of setting up the facilities and the cost of transporting people to the facilities is minimized.

The facility location problems capture the common features of the problems described above. The goal of facility location is to serve a set of demand points, typically called as clients, by a opening a set of access points, typically called as facilities, in a cost effective and efficient manner. The distances between the clients and facilities are assumed to satisfy metric properties. There is a location dependent cost for opening a facility at each location. Typically, a solution to a facility location problem is specified by a set of facilities to be opened and an assignment of the clients to the open facilities. The sum of costs of opening the facilities is called the facility cost, and the sum of distances of each client to the facility it is assigned to is called as the service cost of the solution. Different variants of the facility location problem are obtained by combining these costs in different ways. A richer set of problems emerges by considering formulations in which each facility can serve at most a specified number of clients or by making the cost of a facility depend on the number of clients it serves after the assignment. A comprehensive treatment of facility location problems and their formulations can be found in [15, 40, 45].

It is clear that facility location is a very important aspect of organizational tasks. Naturally, they have been solved for a long time now. The advent of computers fostered interest in obtaining efficient and optimal solutions to these problems. But, most variants of facility location are NP-Complete. So, efficient algorithms which compute solutions which are close to the optimal solution are desirable. In the last few years, approximation algorithms for facility location problems have gained attention of many researchers.

In the past few years, many approaches have been proposed to develop approximation algorithms for facility location problems. Greedy heuristics were the first to be proposed and have been refined ever since. Another popular approach is to consider the linear program relaxation of the integer program formulation, and round an optimal fractional solution so as to guarantee good approximations. Primal-dual algorithms based on the integer program formulations have also been used successfully. Combination of greedy heuristics with primal-dual techniques have obtained very good results for certain varieties of facility location problems. However, techniques based on local search have been used successfully for approximating the largest variety of facility locations problems. For certain variations of the facility location problems, local search gives the best known approximation factor of all the proposed approaches and for certain other variations, local search is the only technique known to give acceptable bounds.

In this thesis, we present local search heuristics for a wide range of facility location problems and analyze their approximation guarantees. We also establish an interesting connection between the price of anarchy of a network service provider game and the locality gap of the corresponding facility location problem. This gives rise to the possibility of reducing the question of upper bounding of the price of anarchy of certain games to the question of upper bounding the locality gap of the corresponding optimization problems.

Chapter 2

Overview

In this chapter, we present an overview of this thesis. In Section 2.1, we define the problems considered in this thesis. In Section 2.2, we present a brief summary of the known techniques for facility location. In Section 2.3, we present an overview of the local search technique for combinatorial optimization problems.

2.1 Facility Location Problems

As described in chapter 1, the set of facility locations or simply, facilities, and the set of clients form a part of the input for all facility location variants. We denote the set of *facilities* by F and the set of *clients* by C. Typically, we are required to *open* a subset of the facilities and serve the demands of clients by assigning them to one of the open facilities. The distance between a pair of points $i, j \in F \cup C$ is denoted by c_{ij}. The distances are symmetric, satisfy triangle inequalities, and $c_{ij} = 0$, if and only if $i = j$. Thus, the distances define a metric. Suppose S is a set of open facilities, then $\forall i \in C$, $dist(i, S)$ denotes the minimum distance between i and a facility in S. If a client j is served by a facility i in a solution, then the distance c_{ij} is said to be the *service cost of client* j. The sum of service costs of all clients is the *service cost* of the solution. Depending on the variant we are dealing with, there is a cost associated with opening a facility at $i \in F$, denoted by f_i. For a given solution, the cost of opening all the open facilities is called its *facility cost*. Certain variants of facility location may also place a limit on the number of clients a facility can serve, namely its *capacity*.

A rich set of facility location problems emerges by mixing the facility costs, service costs, and the capacity constraints differently. As demonstrated in this thesis, local search provides a unified approach to obtain approximation algorithms for these problems.

2.1.1 k-median problem

This problem is motivated by scenarios in which a limited budget is available for opening the facilities and the cost of all the facilities are roughly the same.

- **Input:** The set of facilities F, the set of clients C, and the distance metric. Input also consists of an integer k which is the maximum number of facilities that can be opened.
- **Output:** Find a set $S \subseteq F$ such that $|S| \leq k$, and $\sum_{i \in C} dist(i, S)$ is minimized.

We show that a simple local search heuristic gives a $3(1 + \epsilon)$ approximation for the k-median problem. Currently, this is the best known approximation ratio for the k-median problem.

2.1.2 Uncapacitated Facility Location (UFL)

There are facility location situations in which both the facility cost and service cost are incurred only once. The UFL problem models such scenarios.

- **Input:** The set of facilities F, the set of clients C, and the distance metric. For each $i \in F$, the cost of opening a facility at i, denoted by f_i, is also given.
- **Output:** Find a set $S \subseteq F$ such that $(\sum_{i \in S} f_i + \sum_{i \in C} dist(i, S))$ is minimized.

We provide a local search heuristic with a $3(1 + \epsilon)$ approximation for the UFL problem. Our algorithm considers very simple local operations.

2.1.3 k-Uncapacitated Facility Location (k-UFL)

This problem is a variant of uncapacitated facility location in which a limit is placed on the number of facilities which can be opened.

- **Input:** The set of facilities F, the set of clients C, an integer k, and the distance metric. For each $i \in F$, cost of opening a facility at i, denoted by f_i, is also given.
- **Output:** Find a set $S \subseteq F$ such that $|S| \leq k$ and $(\sum_{i \in S} f_i + \sum_{i \in C} dist(i, S))$ is minimized.

Note that, k-UFL is a generalization of 2.1.1 and 2.1.2. It reduces to the k-median problem if all the facility costs are zero and reduces to the UFL if k is equal to the number of the facilities in F. We [16] give the first known analysis of a local search heuristic for this problem and obtain an approximation factor of $5(1+\epsilon)$. We also establish an interesting connection between the local optimum of this problem and the worst-case equilibria of a related network service provider game.

2.1.4 Facility Location with Soft Capacities (∞-CFL)

The uncapacitated facility location ignores the fact that the cost of a facility could depend on the number of clients it serves. This problem attempts to take this aspect into account by associating capacities with the facilities and making the cost of facility vary linearly with respect to the number of clients it serves modulo its original capacity.

- **Input:** The set of facilities F, the set of clients C, and the distance metric. For each $i \in F$, cost of opening a facility at i denoted by f_i, and a capacity u_i which is the maximum number of clients a facility at i can serve.
- **Output:** A function h from the set of facilities to the set of integers, $h : F \rightarrow \mathbf{N}$, the set of natural numbers. It specifies the number of copies of a facility being opened. Output also consists of an assignment of clients to the set of facilities, $g : C \rightarrow F$. The assignment should be such that $(s_i = |\{j \in C | g(j) = i\}| \leq h(i) \cdot u_i), \forall i \in F$. In other words, we have to open a subset of facilities (multiple copies of a facility can be allowed which increases its capacity by a factor of its original capacity) and assign clients to the open facilities such that no facility serves more than its effective capacity. The goal is to minimize $\sum_{i \in F} h(i) \cdot f_i + \sum_{j \in C} c_{jg(j)}$, i.e, minimize the total cost of opening the facilities and service cost of all clients.

Note that the capacities at the facilities are not uniform. We present the first local search algorithm for ∞-CFL with non-uniform capacities. We show that our algorithm has an approximation ratio of $4(1+\epsilon)$.

2.1.5 Universal Facility Location Problem (UniFL)

In the ∞-CFL problem, the cost of a facility scales linearly with the number of clients it serves module its capacity. A generalization of this feature would be to allow the cost of a facility to be a function of the number of clients it serves. The UniFL problem generalizes the ∞-CFL in this manner.

- **Input.**The set of facilities F, the set of clients C, and the distance metric between them. For each $i \in F$ we are given a function $G_i(.)$ which is assumed to be non-decreasing, and left-continuous mapping from non-negative reals to non-negative reals. G_i represents the cost of the facility i depending on the number of clients it serves, i.e, its load.

- **Output.** Let $S \subseteq F$ be the set of facilities opened and let M be a mapping from clients to S, i.e, $M : C \rightarrow S$. Thus, for $j \in C$, $M(j)$ denotes the facility in S to which j is assigned. For each $i \in S$, n_i denotes the number of clients that i serves , i.e, $n_i = |\{j \in C | M(j) = i\}|$. The goal is to identify a set S and an assignment M of clients to the facilities in S in such that $\sum_{i \in S} G_i(n_i) + \sum_{j \in C} c_{M(j)j}$ is minimized.

In the above formulation, all the clients are assumed to have unit demand. The function G_i is defined to capture when the demands can be arbitrary and demand of a client can be split across facilities. Consider a special case of ∞-CFL in which a single copy of a facility is allowed to be opened. This can be specified by an UniFL instance in which $G_i(x) = f_i$ if $x \leq u_i$ and ∞, otherwise. This version of facility location is called Capacitated Facility Location with Hard Capacities or 1-CFL.

2.1.6 Budget Constrained k-median problem

We introduce the following problem motivated by the contrasting objectives of the k-median and k-center problems. The objective function in k-median problem minimizes the average distance traveled by the clients. In the k-center problem, the goal is to open k facilities such that the maximum distance of any client from its nearest facility is minimized. In many situations, it is desirable to obtain a simultaneous approximation for both the problems. We consider the problem of minimizing the total service cost of when a limit is placed on the maximum service cost that can be incurred by a client.

- **Input:** The set of clients C. The distances between clients in C satisfy metric properties. Input also consists of an integer k and a *budget* B. For a client $j \in C$ and a set $S \subseteq C$, $ds(j, S)$ denotes $\min\limits_{i \in S} c_{ij}$.

- **Validity:** A *valid solution* $S \subseteq C$ is such that, $(|S| \leq k) \wedge (\forall\, i \in C : ds(i, S) \leq B)$.

- **Assumption:** The input has at least one valid solution.

- **Output:** A valid solution S such that $\sum_{i \in C} ds(i, S)$ is minimized.

The nature of the k-median problem and the k-center problem makes it impossible to obtain a solution which is good with respect to both these measures. So, we relax the constraints by allowing the budget on the k-center measure to be exceeded by a constant factor. We say that an algorithm gives (α, β) approximation for this problem if it opens k facilities whose median cost is at most α times the cost of the optimal valid solution while its induced center cost is at most βB. We give a local search algorithm for this problem with a pre-processing stage.

2.2 Techniques for Facility Location

The study of approximation algorithms for facility location began with the work of Hochbaum [23]. Research in the last decade has improved the state of the art dramatically. We give an overview of the different techniques which have been used successfully to approximate some variants of facility location.

2.2.1 Greedy Heuristics

Approximation algorithms based on greedy heuristics were the first to be proposed for facility location problems by Hochbaum [23]. She reduced the facility location problems to variants of set cover and proposed algorithms along the lines of the greedy heuristic for the set cover problem, and proved $O(\log n)$ bounds. Recently, Jain et al. [25] showed that the same algorithm can be shown to provide constant factor approximation for uncapacitated facility location. A modified greedy algorithm for uncapacitated facility location problem was analyzed by Jain et al. [26] using factor revealing LP which exploits the special properties of the heuristic and also the structure of the problem.

2.2.2 LP Rounding Techniques

Approximation algorithms based on rounding the fractional optimal solution to the LP relaxation of the original integer programs were proposed by Shmoys et al. [53]. They used the filtering idea proposed by Lin and Vitter [38] to round the fractional solution to the LP and obtain constant factor approximations for many facility location problems. This idea was also combined with randomization by Chudak and Shmoys [13].

2.2.3 Primal-Dual Techniques

Approximation algorithms for facility location based on primal-dual techniques were proposed by Jain and Vazirani [27]. They solved the uncapacitated facility location problem using a two-phase primal-dual scheme. Their technique's novelty was in relaxing the primal conditions while satisfying all the complimentary slackness conditions. This allowed them to prove a stronger approximation theorem for uncapacitated facility location. This also allowed them to obtain approximation algorithms for a variety of facility location problems including the k-median problem using the Lagrangian relaxation technique.

2.2.4 Local Search Techniques

Approximation algorithms for facility location based on local search are perhaps the most versatile. Local search heuristics have been used for many years by practitioners and one such heuristic was proposed by Kuehn and Hamburger [35]. However, Korupolu et al. [32] showed for the first time that a worst case analysis of the local minimas computed by these heuristics was possible and they showed constant factor approximations to many facility location problems which were comparable to those obtained by other techniques. The significance of these results lies in the fact that local search is routinely implemented by most practitioners of operations research. For certain variants of facility location problems, local search is the only technique known to give constant factor approximations.

2.3 Local Search Technique

In this section, we discuss the local search technique for combinatorial optimization problems. Local search algorithms have been popular among practitioners due to their ease of understanding and implementation. For many optimization problems, local search heuristics are also method of choice for implementation. We begin by briefly surveying the use of local search heuristics in designing algorithms for combinatorial optimization problems.

Local search techniques have been very popular as heuristics for hard combinatorial optimization problems. The 1-exchange heuristic by Lin and Kernighan [39] for the metric-TSP remains the method of choice for practitioners. However, most of these heuristics have poor worst-case guarantees and very few approximation algorithms that rely on local search are known. Fürer and Raghavachari [17] proposed the first non-trivial approximation algorithm based on local search.

They considered local search for the problem of computing spanning trees whose maximum degree is minimum. Their algorithm computed a spanning tree and Steiner trees within an additive logarithmic error of the optimum. It computes a spanning tree whose degree is at most $O(\Delta^* + \log n)$ where Δ^* is the degree of some optimal tree and n is the number of nodes in the input graph. Subsequently, local search was used to design approximation algorithms for degree constrained network design problems. Ravi, Raghavachari, and Klein [49] generalized the above approach to design local search heuristics for the problem of finding one-connected networks that are cut-covers of proper functions such that the maximum degree of any node in the network is minimum. They gave quasi-polynomial ($n^{O(\log_{1+\epsilon} n)}$-time) algorithm which computes solutions whose maximum degree is at most $(1+\epsilon)$ times the minimum with an additive error of $O(\log_{1+\epsilon} n)$, for any $\epsilon > 0$. Lu and Ravi [41] consider local search for the problem of computing the spanning tree with maximum number of leaf nodes. They prove approximation guarantees of 5 and 3 for 1-change and 2-change local search heuristics respectively. Two spanning trees T_1 and T_2 of an n-node graph are said to be distance k apart if they have $(n-1-k)$ edges in common. A k-change heuristic for the above problem considers trees which are at most distance k from the current solution for local improvement. Khanna et al. [29] showed that a simple local search algorithm for a special case of TSP in which all the edge lengths are restricted to be 1 or 2, gives an approximation of 3/2. They also showed that their analysis is tight. Könemann and Ravi [31] used local search algorithms for degree-bounded minimum spanning trees. Chandra et al. [7] show an approximation factor of $4\sqrt{n}$ for the 2-exchange local search heuristic for the Euclidean traveling salesman problem. Khuller et al. [30] give a local search approximation algorithm for finding a feedback edge-set incident upon the minimum number of vertices. Local search has also been used for set packing problems by Arkin and Hassin [2]. The study of local search heuristics for facility location problems began with the work of Korupolu et al. [32] who first showed that the heuristics proposed by Kuehn and Hamburger [35] yield bounds which are comparable to those obtained by other methods.

2.3.1 Local Search and Locality Gap

A generic local search algorithm is shown in Figure 2.1. Consider an optimization problem P and an instance I of the problem. A local search algorithm $LS(P)$ produces a solution to the instance I by iteratively exploring the space of all feasible solutions to I. Formally, the algorithm can be described by the set $\mathcal{S}$ of all feasible solutions to the input instance, a cost function $cost : \mathcal{S} \to I\!\!R$, a *neighborhood* structure $\mathcal{N} : \mathcal{S} \to 2^{\mathcal{S}}$ and an oracle that given any solution $S \in \mathcal{S}$, finds (if possible) a solution $S' \in \mathcal{N}(S)$ such that $cost(S') < cost(S)$. A solution $S \in \mathcal{S}$ is called *local optimum* if $cost(S) \leq cost(S')$ for all $S' \in \mathcal{N}(S)$. The algorithm in Figure 2.1 returns a locally optimum solution. The cost function and the neighborhood structure $\mathcal{N}$ will vary depending on the problem and the heuristic being employed. The neighborhood structure usually specifies the local operations allowed at each step. In case of facility location problems, the algorithm described in Figure 2.1 can be modified suitably to run it in polynomial time and argue approximability. Our description of a local search algorithm is similar to the description given by Yannakakis [57].

```
    Algorithm     Local Search.

1.  $S \leftarrow$ an arbitrary feasible solution in $\mathcal{S}$.
2.  While $\exists\, S' \in \mathcal{N}(S)$ such that $cost(S') < cost(S)$,
        do $S \leftarrow S'$.
3.  return $S$.
```

Figure 2.1: A generic local search algorithm

Consider a minimization problem P and a local search procedure to solve P, denoted by $LS(P)$. For an instance I of the problem P, let `global`(I) denote the cost of the global optimum and `local`(I) be the cost of a locally optimum solution provided by $LS(P)$. We call the supremum of the ratio `local`(I)/`global`(I), the *locality gap* of $LS(P)$.

2.3.2 Analysis Technique

In this section, we present a general framework for running local search heuristics efficiently and to convert the argument for locality gap into an approximation algorithm for the problem.

The generic algorithm shown in Figure 2.1 may not always terminate in polynomial time. To run it polynomial time, we modify step 2 of the algorithm as follows.

```
2M.  While $\exists\, S' \in \mathcal{N}(S)$ such that $cost(S') \leq (1 - \epsilon/Q)\, cost(S)$,
         do $S \leftarrow S'$.
```

Here $\epsilon > 0$ is a constant and Q is a suitable integer which is polynomial in the size of the input. Thus, in each local step, the cost of the current solution decreases by a factor of at least ϵ/Q. If O denotes an optimum solution and S_0 denotes the initial solution, then the number of steps in the algorithm is at most $\log(cost(S_0)/cost(O))/\log \frac{1}{1-\epsilon/Q}$. As $Q, \log(cost(S_0))$, and $\log(cost(O))$

are polynomial in the input size, the algorithm terminates after polynomially many local search steps. We choose Q such that, the algorithm with the above modification continues to have a small locality gap.

We now present a generic technique for proving a bound on the locality gap. If S is a locally optimum solution then for all $S' \in \mathcal{N}(S)$,

$$cost(S') - cost(S) \geq 0.$$

The key to arguing locality gap is to identify a suitable, polynomially large(in the input size) subset $\mathcal{Q} \in \mathcal{N}(S)$ of neighboring solutions which satisfies the following property:

$$\sum_{S' \in \mathcal{Q}} (cost(S') - cost(S)) \leq \alpha \cdot cost(O) - cost(S)$$

where O is an optimum solution and $\alpha > 1$ is a suitable constant. But,
$\sum_{S' \in \mathcal{Q}} (cost(S') - cost(S)) \geq 0$ as S is locally optimum. This implies that $cost(S) \leq \alpha \cdot cost(O)$ and gives a bound of α on the locality gap.

Let us now consider a solution S output by the algorithm after incorporating the modified step 2M (with $Q = |\mathcal{Q}|$). To analyze the quality of S, we note that for all $S' \in \mathcal{Q}$, $cost(S') > (1 - \epsilon/Q) cost(S)$. Hence

$$\alpha \cdot cost(O) - cost(S) \geq \sum_{S' \in \mathcal{Q}} (cost(S') - cost(S)) > -\epsilon \cdot cost(S)$$

which implies that $cost(S) \leq \frac{\alpha}{(1-\epsilon)} cost(O)$. Thus our proof that a certain local search procedure has a locality gap of at most α translates into a $\alpha/(1-\epsilon)$ approximation algorithm with a running time that is polynomial in the input size and $1/\epsilon$.

In certain cases, the local operations allowed at each step may have an exponentially large neighborhood. So, to check if there exists a neighbor which improves the cost, one may have to check all the solutions in the neighborhood. In such cases, we work with polynomial time heuristic which considers only a subset of the neighborhood. So, the algorithm may terminate in a solution which is not local minima in the strict sense. However, finding a set of neighbors $\mathcal{Q} \in \mathcal{N}(S)$ which satisfy the condition specified in equation 2.3.2 is sufficient to prove corresponding approximation results.

2.4 Local Search and Approximation Classes

The study of the relationship between approximability of NP-optimization problems and the quality of local optimums in their local search space was initiated by Yannakakis [57]. It was further pursued by Ausiello and Protasi [4] and Khanna et al. [29]. Here, we present a brief survey of these results.

Consider an NP-optimization problem P and a neighborhood function $\mathcal{N}$ for the problem. For a given instance I of the problem with the set of valid solutions denoted by $\mathcal{S}(I)$, the neighborhood function is given by $\mathcal{N} : \mathcal{S}(I) \rightarrow 2^{|\mathcal{S}(I)|}$. We can associate a distance measure between any two solutions $S_1, S_2 \in \mathcal{S}(I)$; let this be given by $HD(S_1, S_2)$. The neighborhood function $\mathcal{N}$ is said to be of distance d if, $((S_1, S \in \mathcal{S}(I)) \wedge (S_1 \in \mathcal{N}(S)))$ implies that $HD(S_1, S) \leq d$. For a given optimization problem, we are interested in the relationship between the distance of a neighborhood function and the worst-case quality of the local optimum. This study was initiated independently by Ausiello and Protasi [4], and Khanna et al. [29].

2.4.1 Guaranteed Local Optima(GLO)

Ausiello and Protasi [4] gave this useful characterization of an NP-optimization problem. Informally, an NP-optimization problem (maximization) is said to be GLO if and only if it has neighborhood with a constant distance, and constant locality gap. Formally, an NP-optimization problem P is said to be in GLO if and only if it has a constant distance neighborhood $\mathcal{N}$ which satisfies the property: If $S \in \mathcal{S}(I)$ is a local optima with respect to $\mathcal{N}$ and $OPT(I)$ is any optimal solution, it is true that $cost(OPT(I))/cost(S) \leq K$ for some constant K. The closure of GLO, denoted by $\overline{\text{GLO}}$ is the set of all NP-optimization problems which are PTAS-reducible to a problem in GLO.

2.4.2 MAX SNP and Non-oblivious Local Search

MAX SNP is the class of NP-optimization problems which can be expressed in terms of the expression

$$\max_T |\{\vec{x} \in U^k | \phi(P_1, \ldots, P_m, T, \vec{x}\}|$$

where U is a finite universe, $P_1, \ldots, P_m$ are predicates of arity r, T is a solution structure, and ϕ is a boolean expression composed of the predicates, T, and the components of $\vec{x}$. k, m are independent of the input, and $P_1, \ldots, P_m$, and U are dependent on the input. MAX SNP is a syntactic characterization of optimization problems as opposed to APX which is a computational characterization.

Non-oblivious local search is a generalized, and powerful local search technique proposed by Khanna et al. [29]. For MAX SNP problems, it gives better approximation guarantee than standard local search. Informally, standard local search (also called oblivious local search) uses the objective function itself to guide the search of the neighborhood. In contrast, the non-oblivious local search uses a more general cost function to guide the search. Given a structure T, it computes a weighted linear combination of the predicates satisfied by the structure. Formally, given a structure T,

$$\text{cost}(T) = \sum_{\vec{x} \in U^k} \sum_{i=1}^{k_1} p_i \phi_i(P_1, \ldots P_m, T, \vec{x}).$$

Non-oblivious GLO is the set of all NP-optimization problems which can be approximated within constant factor by non-oblivious local search using neighborhood structures of constant distance.

2.4.3 Local Search and Approximation

Ausiello and Protasi [4] studied the relationship between APX class of NP-optimization problems and the natural local search for these problems. APX is the set of NP-optimization problems which can be approximated within a constant factor. Their key contribution was in showing that the set APX is equal to the closure of GLO. The following two lemmas summarize the results obtained by them.

Lemma 2.4.1 (Ausiello and Protasi) *Every problem in* GLO *is in* APX.

Lemma 2.4.2 (Ausiello and Protasi) $\overline{\text{GLO}}$ = APX.

They showed that the Vertex Cover(VC) problem has no constant distance neighborhood with a constant locality gap. Consequently, they also showed that,

Lemma 2.4.3 (Ausiello and Protasi) GLO *is* strict *subset of* APX.

Khanna et al. [29] studied the relationship between MAX SNP and non-oblivious local search. They showed that the traditional oblivious local search is not sufficient to characterize MAX SNP. They also showed that non-oblivious local search is more powerful than oblivious local search. In particular, they showed that non-oblivious local search can characterize the MAX SNP class. Their results are summarized by the following lemmas.

Lemma 2.4.4 (Khanna, Motwani, Sudan, and Vazirani) MAX SNP $\not\subseteq$ GLO.

Lemma 2.4.5 (Khanna, Motwani, Sudan, and Vazirani) GLO *is a* strict *subset of* non-oblivious GLO.

Lemma 2.4.6 MAX SNP $\subseteq$ non-oblivious GLO.

These results show the relevance of local search in complexity theory. The study of local search algorithms for a variety of combinatorial optimization problems can help better characterizations as suggested above. All the problems considered in this thesis belong to the class GLO. We are able to give oblivious local search algorithms with constant approximation ratios.

Chapter 3

The k-median problem

In this chapter, we consider the k-median problem defined in 2.1.1. Informally, the input consists of a set of facilities F, a set of clients C, and an integer k. The distances between the facilities and the clients satisfy metric properties. The objective is to open k facilities from F such that the average distance of a client in C to its closest open facility is minimized. Many practical settings are reasonably abstracted by the k-median problem. We show that a simple local search heuristic proposed by Kuehn and Hamburger [35] has a locality gap of 5. We also show that the local operation considered by Kuehn and Hamburger can be generalized to obtain locality gap of 3.

3.1 Preliminaries

Consider a facility location problem where all the facilities have to be opened inside the same city. It is reasonable to assume that the cost of opening a facility is roughly the same in all the locations. Suppose that there is a fixed budget for opening the facilities. This implies that the total number of facilities that can be opened is bounded by the ratio of budget to the cost of a single facility. Such situations are common in operations research and k-median problem is a reasonable abstraction of these situations.

Kuehn and Hamburger [35] proposed a local search heuristic which considered a simple swap operation at each step. The algorithm starts with an arbitrary subset of k facilities. At each step, it tries to improve the solution by removing one of the facilities from current solution and adding a new facility. The algorithm terminates when the solution cannot be improved in this manner. This is one of the most popular heuristic used in practice. Hochbaum [23] considered greedy heuristics for facility location problems in which the distances need not satisfy metric properties. But, her technique did not yield any approximation guarantee for the k-median problem. Lin and Vitter [38, 37] gave a bicriteria approximation algorithm which, for any $\epsilon > 0$, finds a solution of cost at most $2(1+\epsilon)$ times the optimal which opens at most $(1+1/\epsilon)k$ facilities. The first approximation algorithm which opens at most k facilities was obtained by Bartal [6, 5]. He combined the approximation of any metric with a tree metric with the fact that the k-median problem can be solved optimally on tree metric. In effect, he gave a randomized algorithm with an $O(\log n \log\log n)$ approximation ratio. This algorithm was derandomized and further refined by Charikar et al. [8] who gave an $O(\log k \log\log k)$ approximation bound. The first constant factor approximation for the k-median problem was given by Charikar et al. [10]. They used the filtering idea of Lin and Vitter to round the optimal fractional solution of a linear program relaxation of the integer program formulation to obtain a half-integral solution. They also proposed a heuristic to convert the half-integral solution to an integral solution and proved an approximation factor of 6.66. Jain and Vazirani [27] proposed an approximation algorithm by viewing the UFL problem as Lagrangian relaxation of the k-median problem. They proposed a primal-dual algorithm for the UFL problem which has an approximation factor of 3. Their solution satisfied a stronger property that the algorithm computes a solution whose sum of facility cost and three times the service cost is at most three times the total cost of the optimal solution. They used this algorithm with the Lagrangian relaxation to obtain an approximation ratio of 6 for the k-median problem. Their analysis was improved by Charikar and Guha [9]. They showed that the k-median problem has an approximation ratio of 4. Recently, Archer et al. [1] showed that the LP relaxation of the natural IP formulation has an integrality gap of 3. They modified the primal-dual algorithm for the UFL problem given by Jain and Vazirani to satisfy a "continuity" property and used it to demonstrate the integrality gap. However, their proof gives only an exponential time algorithm. The best known hardness result for the k-median problem was given by Jain et al. [25]. They showed that the k-median problem cannot be approximated within a factor of $(1+2/e-\epsilon)$, for $\epsilon > 0$, unless $NP \subseteq DTIME(n^{O(\log\log n)})$.

The first analysis of a local search heuristic for the k-median problem was given by Korupolu et al. [32, 33]. They considered local operations of adding a facility, dropping a facility and swapping a pair of facilities at each step. They showed that such a local search algorithm gives a $3+5/\epsilon$ approximation and opens at most $k(1+\epsilon)$ facility. So, their result gives a pseudo-approximation to the k-median problem. We analyze local search heuristic for the k-median problem proposed by Kuehn and Hamburger [35]. We [3] prove that its locality gap is 5. We show that this guarantee can be improved by considering stronger local operation. We consider p-swap operation which deletes

p facilities from the current solution and adds p facilities to the current solution. We show that the local search heuristic with p-swaps has a locality gap of $3 + 2/p$. This is the first analysis of a local search heuristic which opens at most k facilities. Our analysis translates to an approximation guarantee of $3(1+\epsilon)$ and is currently the best known. Subsequently, Kanungo et al. [28] considered the same heuristic for the k-means problem where the goal is to identify k facilities such that the sum of squares of distances of clients to their nearest facility is minimized. They showed that, virtually the same analysis yields approximation factors of $25 + \epsilon$ and $9 + \epsilon$ for the single-swap and multiple swaps heuristics respectively.

3.2 Notations

Consider a solution to an instance of the k-median problem. Suppose S is the set of facilities which are opened. The most natural and in fact, the optimal assignment of clients to the facilities in S is : assign each client to the closest facility in S. So, the service cost of each client is well defined once the set S is specified. It is natural to partition clients depending on the facility that serves it in the solution. In fact, these partitions form the Voronoi partitioning of the set of clients. Most of these ideas are relevant for other variants of facility location. In this section, we formalize these notions and also introduce notations used in rest of the chapters. Here, we introduce these notions in the broader context of facility location and it is easy to obtain precise definitions for each problem.

A solution to a typical facility location problem is given by a set of open facilities A, and an assignment of each client to an open facility. Let $\sigma : C \rightarrow A$ denote the assignment function. The choice of facilities and the actual assignments vary depending on the constraints imposed by the problem specification. The most important notions used in the analysis of local search algorithms for facility location are that of neighborhood of a facility, and the service cost of a client. These notions are illustrated in Figure 3.1. The neighborhood of a facility is the set of clients assigned to it. The service cost of a client is its distance from the facility it is assigned to. Formally, for a facility $a \in A$, the neighborhood of a denoted by $N_A(a)$ is defined as $N_A(a) = \{j|\sigma(j) = a\}$. For a subset of facilities, $T \subseteq A$, let $N_A(T) = \bigcup_{a \in T} N_A(a)$. The service cost of a client $j \in C$, denoted by A_j is given by $c_{j\sigma(j)}$. Note that the assignment is straightforward in case of the k-median problem and the UFL problem. Each client is assigned to the nearest facility. However, in case of capacitated versions, these assignments have to be computed by solving appropriate transshipment problems. For any given instance of a problem, we denote an optimal solution by O. A solution computed by a local search algorithm at a step, including the locally optimum solution is denoted by S.

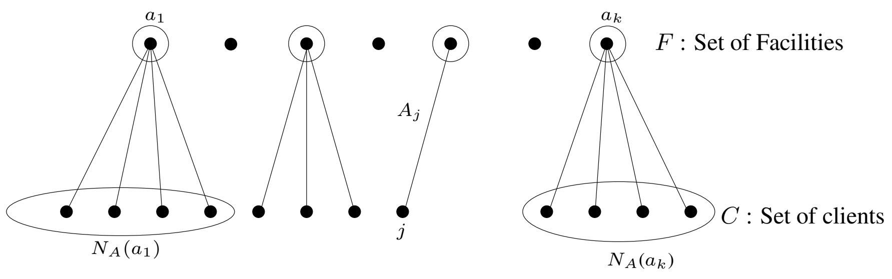


Figure 3.1: Notions of neighborhood of a facility and service cost of a client

3.3 Local search with single swaps

In this section, we consider a local search whose only local operation is a single-swap which improves the cost. A swap is effected by closing a facility $s \in S$ and opening a facility $s' \notin S$ and is denoted by $\langle s, s' \rangle$; hence $\mathcal{N}(S) = \{S - \{s\} + \{s'\} \mid s \in S\}$. We start with an arbitrary set of k facilities and keep improving our solution with such swaps until we reach a locally optimum solution. The algorithm described in Figure 2.1 is reproduced here with suitable modifications. Similar modifications can also be done for local search heuristics for other facility location problems and we do not reproduce them each time. We use $S - s + s'$ to denote $S - \{s\} + \{s'\}$.

```
    Algorithm    Local Search.

1.  Let $S \subseteq F$ be such that $|S| = k$.
2.  While $\exists s \in S, s' \in F$ s.t. $cost(S - \{s\} + \{s'\}) < cost(S)$,
        do $S \leftarrow S - \{s\} + \{s'\}$.
3.  return $S$.
```

Figure 3.2: Local Search heuristic for the k-median problem

3.3.1 The analysis

We now show that this local search procedure has a locality gap of 5. As always, S denotes the locally optimum solution output by the local search procedure and O denotes an optimum solution.

From the local optimality of S, we know that,

$$cost(S - s + o) \geq cost(S) \quad \text{for all } s \in S, o \in O. \tag{3.1}$$

Note that even if $S \cap O \neq \emptyset$, the above inequalities hold. We combine these inequalities judiciously to show that $cost(S) \leq 5 \cdot cost(O)$.

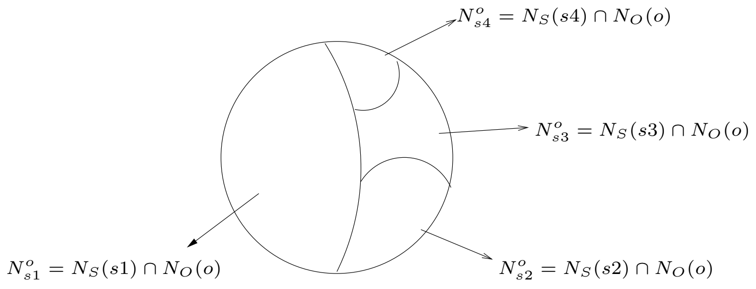

$N^o_{s4} = N_S(s4) \cap N_O(o)$
$N^o_{s3} = N_S(s3) \cap N_O(o)$
$N^o_{s1} = N_S(s1) \cap N_O(o)$
$N^o_{s2} = N_S(s2) \cap N_O(o)$

Figure 3.3: Partitioning the neighborhood of a facility $o \in O$

We consider the following partitioning of the neighborhood of a facility $o \in O$ with repsect to the solution S. The partitioning of $N_O(o)$ is such that, each partition consists of all the clients in $N_O(o)$ which are served by a unique facility $s \in S$. Formally, we partition $N_O(o)$ into subsets $N^o_s = N_O(o) \cap N_S(s)$ for all $s \in S$ as shown in Figure 3.3. This definition of N^o_s is used in the analysis of UFL, and ∞-CFL as well.

The main idea in our proof is to consider suitable mapping between the clients in the neighborhood of each optimum facility, and use the mapping to amortize the cost of all the swaps considered. Let us first consider a special case of our analysis to understand the main ideas.

Let us assume that, given a neighborhood $N_O(o)$, we can find a 1-1 and onto function π such that every client in $N_O(o)$ is mapped to a client in $N_O(o)$ which belongs to a partition different from the one that it belongs to. Formally, a client $j \in N^o_s$ is mapped to a client $j' \in N^o_{s'}$ such that $s \neq s'$. Let us consider an arbitrary pairing of the facilities in O with the facilities in S. Specifically, let $\langle s_1, o_1 \rangle \ldots, \langle s_k, o_k \rangle$ be the pairings. We consider the k swaps given by swapping s_i with o_i. In this setting, we show the main ideas of our proof.

When a facility $s \in S$ is swapped with a facility $o \in O$, the clients in the neighborhood of s given by $N_S(s)$ have to be reassigned. We use a specific reassignment to bound the change in cost as a result of the swap. We use the mapping π to reassign the clients in $N_S(s) \cup N_O(o)$ as shown in Figure 3.7. A client $j \in N_O(o)$ is reassigned to o. The change in cost due to this reassignment is $O_j - S_j$. A client $j \in N_S(s) \setminus N_s^o$ is reassigned as follows. Let $\pi(j) \in N_{s'}^o$. By our assumption, $s' \neq s$. We reassign j to s'. By triangle inequality, the change in cost due to this reassignment is bounded by $O_j + O_{\pi(j)} + S_{\pi(j)} - S_j$. The overall change in cost due to any of these swaps is greater than zero as S is a local optimum. The change in cost over all the k swaps defined above gives rise to the following equation,

$$\sum_{i \in \{1,\ldots,k\}} \left(\sum_{j \in N_O(o_i)} (O_j - S_j) + \sum_{j \in N_S(s_i) \setminus N_{s_i}^{o_i}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \right) \geq 0. \tag{3.2}$$

Note that, $\underset{i \in \{1,\ldots,k\}}{\cup} N_O(o_i) = C$. Also, $O_j + O_{\pi(j)} + S_{\pi(j)} - S_j \geq 0$ for all $j \in C$. So, the set $N_S(s_i) \setminus N_{s_i}^{o_i}$ can be replaced by $N_S(s_i)$. Hence,

$$\sum_{j \in C} (O_j - S_j) + \sum_{i \in \{1,\ldots,k\}} \left(\sum_{j \in N_S(s_i)} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \right) \geq 0. \tag{3.3}$$

The first term in the above equation is equal to $cost(O) - cost(S)$. Also, note that $\underset{i \in \{1,\ldots,k\}}{\cup} N_S(s_i) = C$. Thus,

$$cost(O) - cost(S) + \sum_{j \in C} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \geq 0. \tag{3.4}$$

As the mapping π is 1-1 and onto, $\sum_{j \in C} (S_{\pi(j)} - S_j) = 0$, and $\sum_{j \in C} (O_j + O_{\pi(j)})$ is equal to $2 \cdot cost(O)$. Thus,

$$3 \cdot cost(O) \geq cost(S).$$

Observe that the 1-1 and onto mapping function π is very crucial in amortizing the cost over all the k swaps. However, the correctness of the proof is also dependent on the existence of such a mapping function. It is easy to see that such a mapping function does not exist if there are facilities $s \in S$, and $o \in O$ such that $|N_s^o| > \frac{1}{2}|N_O(o)|$. This leads us to categorize the facilities in S based on whether they serve more than half the clients of at least one facility in the optimal solution O. We show the existence of a slightly different mapping function which can be used for reassignment and show a locality gap of 5.

Definition 3.3.1 *We say that a facility $s \in S$* captures *a facility $o \in O$ if s serves more than half the clients served by o, that is, $|N_s^o| > \frac{1}{2}|N_O(o)|$.*

It is easy to see that a facility $o \in O$ is captured by at most one facility in S. We call a facility $s \in S$ *bad*, if it captures some facility $o \in O$, and *good* otherwise. Intuitively, when a facility $s \in S$ capturing a facility $o \in O$ is considered in a swap with a facility $o' \in O$ such that $o' \neq o$, it is difficult to bound the change in service cost of the clients in N_s^o. So, in order to help us in considering the appropriate set of swaps for the analysis, we categorize the facilities in S as good and bad as defined above. Fix a facility $o \in O$ and consider a 1-1 and onto function $\pi : N_O(o) \to N_O(o)$ satisfying the following property (Figure 3.4).

Property 3.3.1 *If s does not capture o, that is, $|N_s^o| \leq \frac{1}{2}|N_O(o)|$, then $\pi(N_s^o) \cap N_s^o = \emptyset$.*

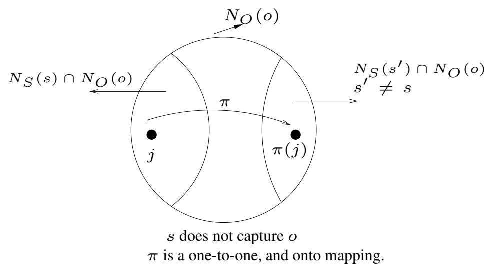


Figure 3.4: The mapping π on $N_O(o)$

We outline how to obtain one such mapping π. Let $D = |N_O(o)|$. Order the clients in $N_O(o)$ as $c_0, \ldots, c_{D-1}$ such that for every $s \in S$, the clients in N_s^o are consecutive, that is, there exists p, q, $0 \leq p \leq q \leq D-1$ such that $N_s^o = \{c_p, \ldots, c_q\}$. Now, define $\pi(c_i) = c_j$ where $j = (i + \lfloor D/2 \rfloor)$ modulo D. For contradiction assume that both $c_i, \pi(c_i) = c_j \in N_s^o$ for some s where $|N_s^o| \leq D/2$. If $j = i + \lfloor D/2 \rfloor$, then $|N_s^o| \geq j - i + 1 = \lfloor D/2 \rfloor + 1 > D/2$. If $j = i + \lfloor D/2 \rfloor - D$, then $|N_s^o| \geq i - j + 1 = D - \lfloor D/2 \rfloor + 1 > D/2$. In both cases we have a contradiction and hence function π satisfies property 3.3.1.

The notion of *capture* can be used to construct a bipartite graph $H = (S, O, E)$ (Figure 3.5). For each facility in S, we have a vertex on the S-side and for each facility in O, we have a vertex on the O-side. We add an edge between $s \in S$ and $o \in O$ if s captures o. It is easy to see that each vertex on the O-side has degree at most one, while vertices on the S-side can have degree up to k. We call H the *capture graph*.

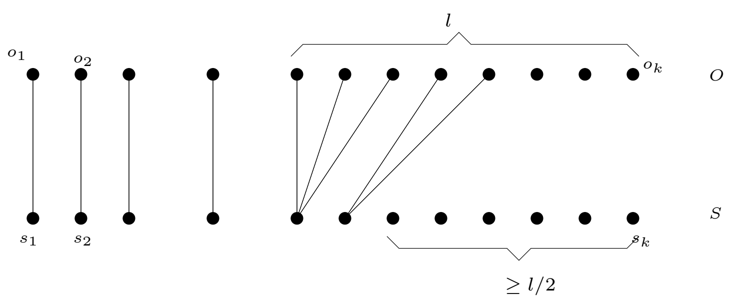


Figure 3.5: Capture Graph $H = (S, O, E)$

Consider a facility $s \in S$ which captures a facility $o \in O$. Suppose s is swapped with a facility $o' \in O$ such that $o' \neq o$. By the definition of the mapping function π, it cannot be used to reassign the clients in N_s^o. Thus, s is constrained to be swapped out with o. Suppose s captures two optimum facilities o_1, o_2. Clearly, s is constrained to be swapped with o_1 or o_2. When s is swapped with one of them, say o_1, the clients in $N_s^{o_2}$ cannot be reassigned using the mapping function π. It is easy to see that, this adds additional constraint that any facility in S which captures two or more facilities in the optimum cannot be involved in any swap.

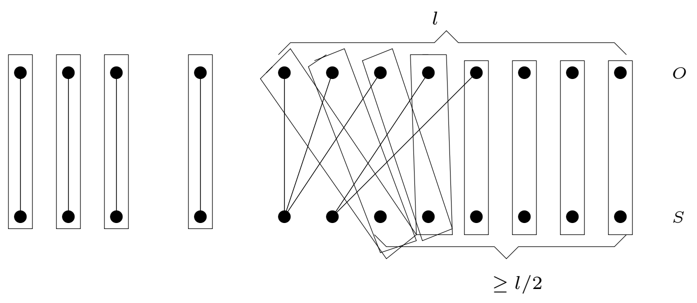


Figure 3.6: k swaps considered in the analysis

We now consider k swaps, one for each facility in O. If some bad facility $s \in S$ captures exactly one facility $o \in O$ then we consider the swap $\langle s, o \rangle$. Suppose l facilities in S (and hence l facilities in O) are not considered in such swaps. Each facility out of these l facilities in S is either good or captures at least two facilities in O. Hence there are at least $l/2$ good facilities in S. Now, consider l swaps in which the remaining l facilities in O get swapped with the good facilities in S such that each good facility is considered in at most two swaps. The bad facilities which capture at least two facilities in O are not considered in any swaps. Figure 3.6 shows the k swaps considered in our analysis. The swaps considered above satisfy the following properties.

1. Each $o \in O$ is considered in exactly one swap.

2. A facility $s \in S$ which captures more than one facility in O is not considered in any swap.

3. Each good facility $s \in S$ is considered in at most two swaps.

4. If swap $\langle s, o \rangle$ is considered then facility s does not capture any facility $o' \neq o$.

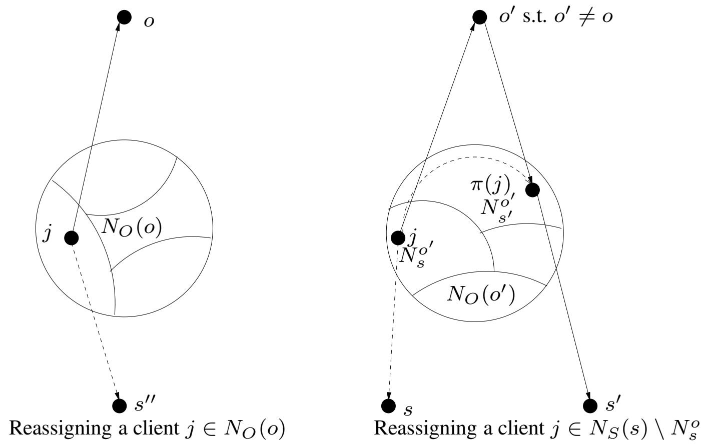


Figure 3.7: Reassigning the clients in $N_S(s) \cup N_O(o)$.

We now analyze each of these swaps. Consider a swap $\langle s, o \rangle$. We place an upper bound on the increase in the cost due to this swap by reassigning the clients in $N_S(s) \cup N_O(o)$ to the facilities in $S - s + o$ as follows (Figure 3.7). The clients $j \in N_O(o)$ are now assigned to o. Consider a client $j \in N_s^{o'}$, for $o' \neq o$. As s does not capture o', by property 3.3.1 of π, we have that $\pi(j) \notin N_S(s)$. Let $\pi(j) \in N_S(s')$. Note that the distance that the client j travels to the nearest facility in $S - s + o$ is at most $c_{js'}$. From triangle inequality, $c_{js'} \leq c_{jo'} + c_{\pi(j)o'} + c_{\pi(j)s'} = O_j + O_{\pi(j)} + S_{\pi(j)}$. The clients which do not belong to $N_S(s) \cup N_O(o)$ continue to be served by the same facility. From inequality (3.1) we have,

$$cost(S - s + o) - cost(S) \geq 0.$$

Therefore,

$$\sum_{j \in N_O(o)} (O_j - S_j) + \sum_{\substack{j \in N_S(s), \\ j \notin N_O(o)}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \geq 0. \tag{3.5}$$

As each facility $o \in O$ is considered in exactly one swap, the first term of inequality (3.5) added over all k swaps gives exactly $cost(O) - cost(S)$. For the second term, we will use the fact that each $s \in S$ is considered in at most two swaps. Since S_j is the shortest distance from client j to a facility in S, using triangle inequality we get: $O_j + O_{\pi(j)} + S_{\pi(j)} \geq S_j$. Thus the second term of inequality (3.5) added over all k swaps is no greater than $2\sum_{j \in C}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j)$. But since π is a 1-1 and onto mapping, $\sum_{j \in C} O_j = \sum_{j \in C} O_{\pi(j)} = cost(O)$ and $\sum_{j \in C}(S_{\pi(j)} - S_j) = 0$. Thus, $2\sum_{j \in C}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) = 4 \cdot cost(O)$. Combining the two terms we get, $cost(O) - cost(S) + 4 \cdot cost(O) \geq 0$. Thus, we have the following theorem.

Theorem 3.3.1 *A local search procedure for the metric k-median problem with the local neighborhood structure defined by, $\mathcal{N}(S) = \{S - \{s\} + \{s'\} \mid s \in S\}$ has a locality gap of at most 5.*

The above algorithm and analysis extend very simply to the case when the clients $j \in C$ have arbitrary demands $d_j \geq 0$ to be served.

3.3.2 Local search with multi-swaps

In this section, we generalize the algorithm in Section 3.3 to consider multi-swaps in which up to $p > 1$ facilities could be swapped simultaneously. The neighborhood structure is now defined by

$$\mathcal{N}(S) = \{(S \setminus A) \cup B \mid A \subseteq S,\ B \subseteq F, \text{ and } |A| = |B| \leq p\}. \tag{3.6}$$

The neighborhood captures the set of solutions obtainable by deleting a set of at most p facilities A and adding set of facilities B where $|B| = |A|$; this swap will be denoted by $\langle A, B \rangle$. We prove that the locality gap of the k-median problem with respect to this operation is exactly $(3 + 2/p)$.

3.3.3 Analysis

We extend the notion of capture as follows. For a subset $A \subseteq S$, we define,

$$capture(A) = \{o \in O \mid |N_S(A) \cap N_O(o)| > |N_O(o)|/2\}.$$

It is easy to observe that if $X, Y \subseteq S$ are disjoint then $capture(X)$ and $capture(Y)$ are disjoint and if $X \subset Y$ then $capture(X) \subseteq capture(Y)$. We now partition S into sets $A_1, \ldots, A_r$ and O into sets $B_1, \ldots, B_r$ such that

1. For $1 \leq i \leq r-1$, we have $|A_i| = |B_i|$ and $B_i = capture(A_i)$; since $|S| = |O|$, it follows that $|A_r| = |B_r|$.
2. For $1 \leq i \leq r-1$, the set A_i has exactly one bad facility.
3. The set A_r contains only good facilities.

A procedure to obtain such a partition is given in Figure 3.8.

```
procedure Partition;
        $i = 0$
        while $\exists$ a bad facility in $S$ do
            1. $i = i + 1$                                        $\{$iteration $i\}$
            2. $A_i \leftarrow \{b\}$ where $b \in S$ is any bad facility
            3. $B_i \leftarrow capture(A_i)$
            4. while $|A_i| \neq |B_i|$ do
                4.1. $A_i \leftarrow A_i \cup \{g\}$ where $g \in S \setminus A_i$ is any good facility
                4.2. $B_i \leftarrow capture(A_i)$
            5. $S \leftarrow S \setminus A_i$
               $O \leftarrow O \setminus B_i$
        $A_r \leftarrow S$
        $B_r \leftarrow O$
end.
```

Figure 3.8: A procedure to define the partitions

Claim 3.3.1 *The procedure defined in Figure 3.8 terminates with partitions of S and O, satisfying the properties listed above.*

Proof. The condition in the while loop in step 4 and the assignment in step 5 of the procedure maintains the invariant that $|S| = |O|$. The steps 3 and 4.2 of the procedure ensure that for $1 \leq i \leq r-1$, we have $B_i = capture(A_i)$ and steps 2 and 4.1 ensure that each for $1 \leq i \leq r-1$, the set A_i has exactly one bad facility. Now before each execution of the step 4.1, we have $|A_i| < |B_i|$. This together with the invariant that $|S| = |O|$ implies that in step 4.1, we can always find a good facility in $S \setminus A_i$. Since with each execution of the while loop in step 4 the size of A_i increases, the loop terminates. The condition in step 4 then ensures that for $1 \leq i \leq r-1$, we have $|A_i| = |B_i|$. Since there are no bad facilities left when the procedure comes out of the outer while loop, we have that the set A_r contains only good facilities. ∎

We now use this partition of S and O to define the swaps we would consider for our analysis. We also associate a positive real weight with each such swap.

1. If $|A_i| = |B_i| \leq p$ for some $1 \leq i \leq r$, then we consider the swap $\langle A_i, B_i \rangle$ with weight 1. From the local optimality of S we have

$$cost((S \setminus A_i) \cup B_i) - cost(S) \geq 0.$$

   Note that even if $A_i \cap B_i \neq \emptyset$ or $S \cap B_i \neq \emptyset$, the above inequality continues to hold.

2. If $|A_i| = |B_i| = q > p$, we consider all possible swaps $\langle s, o \rangle$ where $s \in A_i$ is a good facility and $o \in B_i$. Note that if $i \neq r$, there are exactly $q-1$ good facilities in A_i and for $i = r$, we select any $q-1$ out of the q good facilities in A_r. We associate a weight of $1/(q-1)$ with each of these $q(q-1)$ swaps. For each such swap $\langle s, o \rangle$, we have,

$$cost(S - s + o) - cost(S) \geq 0.$$

Note that any good facility in A_i is considered in swaps of total weight at most $q/(q-1) \leq (p+1)/p$. The swaps we have considered and the weights we assigned to them satisfy the following properties.

1. For every facility $o \in O$, the sum of weights of the swaps $\langle A, B \rangle$ with $o \in B$, is exactly one.

2. For every facility $s \in S$, the sum of weights of the swaps $\langle A, B \rangle$ with $s \in A$, is at most $(p+1)/p$.

3. If a swap $\langle A, B \rangle$ is considered, then $capture(A) \subseteq B$.

For each facility $o \in O$, we partition $N_O(o)$ as follows.

1. For $|A_i| \leq p, 1 \leq i \leq r$, let $N^o_{A_i} = N_S(A_i) \cap N_O(o)$ be a set in the partition.
2. For $|A_i| > p, 1 \leq i \leq r$ and all $s \in A_i$ let $N^o_s = N_S(s) \cap N_O(o)$ be a set in the partition.

As before, for each facility $o \in O$, we consider a one-to-one and onto mapping $\pi : N_O(o) \rightarrow N_O(o)$ with the following property.

Property 3.3.2 *For all sets, P, in the partition of $N_O(o)$ for which $|P| \leq \frac{1}{2}|N_O(o)|$, we have, $\pi(P) \cap P = \emptyset$.*

Such a mapping π can be defined in a manner identical to the one described in Section 3.3.1. The analysis is similar to the one presented for the single-swap heuristic. For each of the swaps defined above, we upper bound the increase in the cost by reassigning the clients. Property 3.3.2 ensures that the function π can be used to do the reassignment as described in Section 3.3.1. We take a weighted sum of the inequalities corresponding to each of the swaps considered above. Recall that in the single swap analysis, we used the fact that each facility in S was considered in at most 2 swaps and upper-bounded the second term of equation (3.5) by $2\sum_{j\in C}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) = 4\cdot cost(O)$. Similarly, we can now make use of the fact that each facility in S is considered in swaps with total weight at most $(p+1)/p$ and upper-bound the second term by $(p+1)/p \cdot \sum_{j\in C}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) = 2(p+1)/p \cdot cost(O)$. This gives us a locality gap of $1 + 2(p+1)/p = 3 + 2/p$.

3.3.4 Tight example

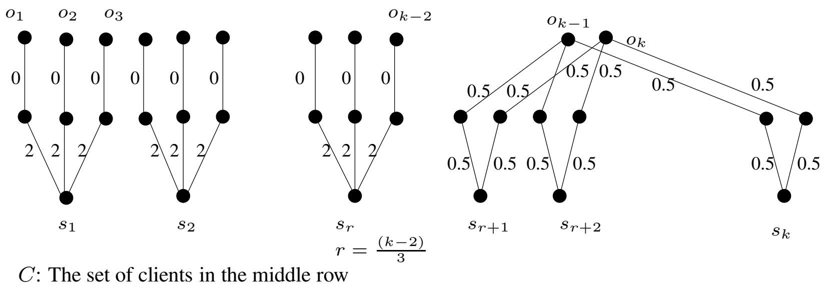


Figure 3.9: Tight example for 2-swap heuristic

In Figure 3.9, we show an instance of the k-median problem in which a solution that is locally optimum for the 2-swap heuristic ($p = 2$) has cost at least $4 - o(1)$ times the cost of the global optimum. Since $3 + 2/p = 3 + 2/2 = 4$ is also the locality gap proved, it shows that the analysis of the 2-swap heuristic is tight. This tight example can be generalized for p-swaps for any $p \geq 1$. In Figure 3.9, the optimal solution is given by $O = \{o_1, o_2, \ldots, o_k\}$, and the solution $S = \{s_1, s_2, \ldots, s_k\}$ is a local optimum. C is the set of clients which are placed in the middle row. In the graph in Figure 3.9, each edge has a label which is its length. The cost of serving a client j by a facility i is length of the shortest path between client j and facility i in the graph; the cost is infinite if there is no path.

Note that $cost(S) = \frac{8k-10}{3}, cost(O) = \frac{2k+2}{3}$ and hence the ratio $\frac{cost(S)}{cost(O)}$ approaches 4 as k approaches ∞. We now show that S is a locally optimum solution, that is, if we swap $\{o_l, o_m\}$ for $\{s_i, s_j\}$ then the cost does not decrease for any choice of i, j, l, and m. To show this, we consider all the possible cases.

1. $i, j \leq r$. Then o_l, o_m will have to lie in the connected components containing s_i, s_j. But in this case the cost would increase by 4.

2. $i \leq r < j$. At least one of o_l, o_m would have to lie in the connected component containing s_i; let this be o_l. If o_m also lies in this component then the cost remains unchanged. If o_m is in a different component and $m \leq k - 2$ then the cost increases by 2. If $m > k - 2$ then the cost of the solution increases by 3.

3. $i, j > r$. If both l, m are at most $k - 2$ then the cost of the solution remains unchanged. The cost remains unchanged even if $l \leq k - 2 < m$. If both l, m are larger than $k - 2$ then, once again, the cost of the solution remains unchanged.

Chapter 4

The uncapacitated facility location problem

In this chapter, we consider uncapacitated facility location and capacitated facility location with non-uniform soft capacities defined in Sections 2.1.2 and 2.1.4 respectively. We analyze local search heuristics for these problems.

4.1 Uncapacitated Facility Location Problem

Uncapacitated facility location is one of the most extensively investigated facility location problems. As defined in Section 2.1.2, the input consists of a set of facilities F, a set of clients C, and distance metric between them. The input also consists of facility cost for each facility in F. Our goal is to open facilities such that sum of facility cost and service cost is minimized. The UFL formulation abstracts the situations in which both the facility cost and service cost are incurred just once. The problem of organizing training camps for a large group of people is one such example.

4.1.1 Preliminaries

Hochbaum [23] gave the first approximation algorithm for UFL based on the greedy heuristic for the set cover problem and proved an approximation factor of $O(\log n)$. She considered the more general case where the distances need not necessarily satisfy triangle inequalities. Subsequently, the approximation factor for UFL has been refined successively using a combination of many techniques. The first constant factor approximation for UFL was given by Shmoys et al. [53]. They used the filtering idea due to Lin and Vitter [38] to round the optimal fractional solution of linear program relaxation of the integer program formulation of the UFL problem. They proved an approximation ratio of 3.16. Guha and Khuller [21] combined this idea with a greedy heuristic to improve the approximation ratio to 2.408. Chudak [12] improved this ratio to $(1 + 2/e)$ by combining the idea of [53]. with the idea of randomization. Jain et al. [25] showed that the greedy algorithm given by Hochbaum gives an approximation factor of 1.861. They gave another greedy algorithm and analyzed it using factor revealing LP to prove an approximation factor of 1.61. Factor revealing LP is a linear program to upper bound the approximation ratio of the greedy heuristic. It is written by exploiting the structural properties satisfied by the greedy algorithm. Mahdian et al. [43] combined these ideas with the idea of scaling which exploits different guarantees on the facility and service cost to give the best known approximation ratio of 1.52. Jain and Vazirani [27] gave a primal-dual algorithm which could be used for approximating k-median problem. The best known hardness result for the UFL problem was given by Guha and Khuller [21]. They showed that the UFL problem cannot approximated with a factor of $(1.463 - \epsilon)$, for $\epsilon > 0$, unless $NP \subseteq DTIME(n^{O(\log \log n)})$.

First local search based approximation algorithm was given by Korupolu et al. [32]. They showed that a local search algorithm which considers the local operation of adding a facility, dropping a facility, and swapping a pair of facilities has a locality gap of 5. Charikar and Guha [9] used a powerful operation of adding a facility and dropping many facilities and showed a locality gap of 3. We [3] show that the algorithm considered in [32] has a locality gap of 3. So, our analysis shows that the bound achieved by Charikar and Guha can be achieved with simpler local operations. Our analysis of the algorithm is tight.

In the UFL problem, we are allowed to open any number of facilities. The goal is to minimize the sum of facility cost and the total service cost. As in the case of the k-median problem, the assignment of clients to the facilities is straightforward and forms Voronoi partitions. Formally, we want to identify a subset $S \subseteq F$ and assign clients in C to facilities in S. Let the assignment function be specified by σ. The goal is to minimize $cost(S) = \sum_{i \in S} f_i + \sum_{j \in C} c_{\sigma(j)j}$.

4.1.2 A local search procedure

We present a local search procedure for the metric uncapacitated facility location problem with a locality gap of 3. The operations allowed in a local search step are adding a facility, deleting a facility, and swapping facilities. Hence the neighborhood $\mathcal{N}$ is defined by

$$\mathcal{N}(S) = \{S + \{s'\} \mid s' \in F\} \cup \{S - \{s\} \mid s \in S\} \cup \{S - \{s\} + \{s'\} \mid s \in S, s' \in F\}. \quad (4.1)$$

4.1.3 The analysis

For any set of facilities $S' \subseteq F$, let $cost_f(S') = \sum_{i \in S'} f_i$ denote the facility cost of the solution S'. Let $cost_s(S')$ denote the service cost of the solution S'. Service cost of the solution S' is given by the total cost of connecting the clients in C to their closest facility in S'. Let $cost(S')$ denote the total cost of the solution S'. Total cost of the solution S' is given by the sum of its facility cost and service cost. These notations are also used in the capacitated versions of facility location. As always, S denotes a local optimum solution and O denotes a global optimum solution. We use the same notation for service cost of a client and the neighborhood of a facility as defined in Section 3.2. We also use the notion of partitioning the neighborhood of an optimum facility as defined in Section 3.3.1. The following bound on the service cost of S was earlier proved in [32].

Lemma 4.1.1 (Service cost)

$$cost_s(S) \leq cost_f(O) + cost_s(O).$$

Proof. Consider an operation in which a facility $o \in O$ is added. Assign all the clients $N_O(o)$ to o. From the local optimality of S we get, $f_o + \sum_{j \in N_O(o)}(O_j - S_j) \geq 0$. Note that even if $o \in S$, this inequality continues to hold. If we add such inequalities for every $o \in O$, we get the desired inequality. ∎

The following lemma gives a bound on the facility cost of S.

Lemma 4.1.2 (Facility cost)

$$cost_f(S) \leq cost_f(O) + 2 \cdot cost_s(O).$$

Proof. As in the case of the k-median problem, we assume that, for a fixed $o \in O$, the mapping $\pi : N_O(o) \to N_O(o)$ is 1-1 and onto, and satisfies property 3.3.1. In addition, we assume that if $|N_s^o| > \frac{1}{2}|N_O(o)|$ then for all $j \in N_s^o$ for which $\pi(j) \in N_s^o$, we have that $\pi(j) = j$. Here we give an outline of how to define such a function π. Let $|N_s^o| > \frac{1}{2}|N_O(o)|$. We pick any $|N_s^o| - |N_O(o) \setminus N_s^o|$

clients j from N_s^o and set $\pi(j) = j$. On the remaining clients in $N_O(o)$, the function π is defined in the same manner as in Section 3.3.1.

Recall that a facility $s \in S$ is *good* if it does not capture any o, that is, for all $o \in O$, $|N_s^o| \leq \frac{1}{2}|N_O(o)|$. The facility cost of good facilities can be bounded easily as follows (see Figure 4.1). Consider an operation in which a good facility $s \in S$ is dropped. Let $j \in N_S(s)$ and $\pi(j) \in N_S(s')$. As s does not capture any facility $o \in O$, we have that $s' \neq s$. If we assign j to s' then we have $-f_s + \sum_{j \in N_S(s)}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \geq 0$. Since for all $j \in N_S(s), \pi(j) \neq j$, the term $\sum_{\substack{j \in N_S(s) \\ \pi(j)=j}} O_j$ is trivially zero and hence we can rewrite the above inequality as

$$-f_s + \sum_{\substack{j \in N_S(s) \\ \pi(j) \neq j}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) + 2 \sum_{\substack{j \in N_S(s) \\ \pi(j)=j}} O_j \geq 0. \tag{4.2}$$

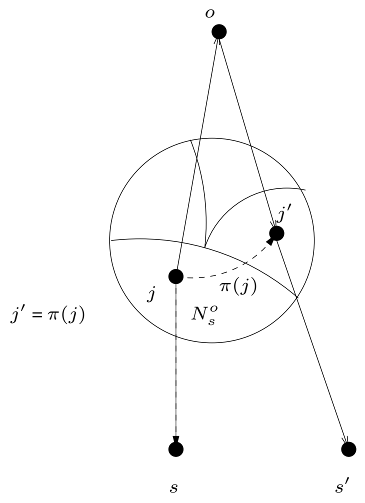


Figure 4.1: Reassigning a client $j \in N_S(s)$ when a good facility s is dropped.

For bounding the facility cost of a bad facility $s \in S$, we proceed as follows. Fix a bad facility $s \in S$. Suppose s captures the facilities $P \subseteq O$. Let $o \in P$ be the facility nearest to s. We consider the swap $\langle s, o \rangle$. The clients $j \in N_S(s)$ are now assigned to the facilities in $S - s + o$ as follows.

1. Suppose $\pi(j) \in N_S(s')$ for $s' \neq s$. Then, j is assigned to s'. Let $j \in N_O(o')$. We have, $c_{js'} \leq c_{jo'} + c_{\pi(j)o'} + c_{\pi(j)s'} = O_j + O_{\pi(j)} + S_{\pi(j)}$ (Figure 4.2(a)).

2. Suppose $\pi(j) = j \in N_S(s)$ and $j \in N_O(o)$. Then, j is assigned to o.

3. Suppose $\pi(j) = j \in N_S(s)$ and $j \in N_O(o')$ for $o' \neq o$. By property 3.3.1 of the mapping π, facility s captures facility o' and hence $o' \in P$. The client j is now assigned to facility o. From triangle inequality, $c_{jo} \leq c_{js} + c_{so}$. Since $o \in P$ is the closest facility to s, we have $c_{so} \leq c_{so'} \leq c_{js} + c_{jo'}$. Therefore, $c_{jo} \leq c_{js} + c_{js} + c_{jo'} = S_j + S_j + O_j$ (Figure 4.2(b)).

Thus, for the swap $\langle s, o \rangle$, we get the following inequality.

$$\begin{aligned} f_o - f_s + \sum_{\substack{j \in N_S(s) \\ \pi(j) \neq j}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \\ + \sum_{\substack{j \in N_O(o), \\ \pi(j) = j \in N_S(s)}} (O_j - S_j) + \sum_{\substack{j \notin N_O(o), \\ \pi(j) = j \in N_S(s)}} (S_j + S_j + O_j - S_j) \geq 0. \end{aligned} \tag{4.3}$$

Now consider an operation in which a facility $o' \in P - o$ is added (Figure 4.2(c)). The clients $j \in N_O(o')$ for which $\pi(j) = j \in N_S(s)$, are now assigned to the facility o' and this yields the following inequality.

$$f_{o'} + \sum_{\substack{j \in N_O(o') \\ \pi(j) = j \in N_S(s)}} (O_j - S_j) \geq 0 \qquad \text{for each } o' \in P - o. \tag{4.4}$$

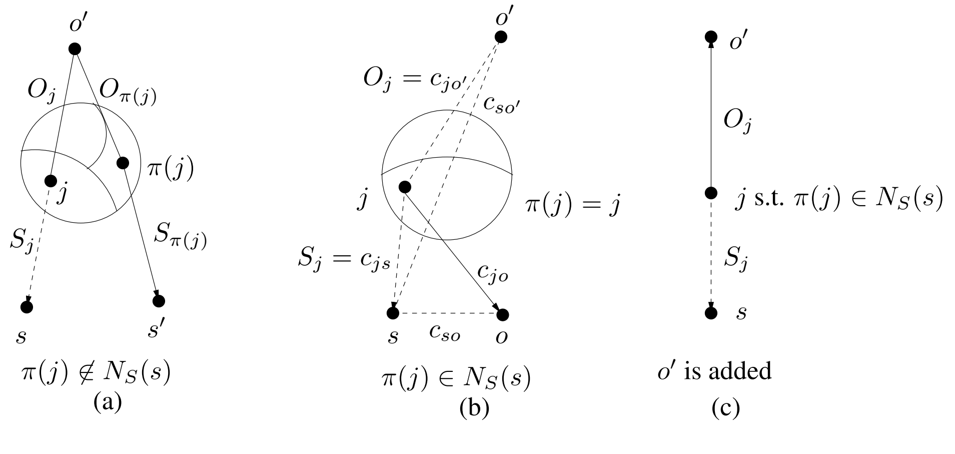


Figure 4.2: Bounding the facility cost of a bad facility s: (a) Reassignment when $\pi(j) \notin N_S(s)$ (b) Reassignment when $\pi(j) \in N_S(s)$ and $j \notin N_O(o)$ (c) Reassignment of $j \in N_O(o')$ when $o' \in P - \{o\}$ is added.

Adding inequality (4.3) with inequalities (4.4) we get, for a bad facility $s \in S$,

$$\sum_{o' \in P} f_{o'} - f_s + \sum_{\substack{j \in N_S(s), \\ \pi(j) \neq j}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) + 2 \sum_{\substack{j \in N_S(s), \\ \pi(j) = j}} O_j \geq 0. \tag{4.5}$$

The last term on the left is an upper bound on the sum of the last two terms on the left of the inequality (4.3) and the last term on the left of the inequality (4.4) added for all $o' \in P - o$.

Now, we add inequalities (4.2) for all good facilities $s \in S$, inequalities (4.5) for all bad facilities $s \in S$ and inequalities $f_o \geq 0$ for all $o \in O$ which are not captured by any $s \in S$ to obtain

$$\sum_{o \in O} f_o - \sum_{s \in S} f_s + \sum_{\pi(j) \neq j} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) + 2 \sum_{\pi(j) = j} O_j \quad \geq \quad 0.$$

Note that $\sum_{j:\pi(j)\neq j} O_j = \sum_{j:\pi(j)\neq j} O_{\pi(j)}$ and $\sum_{j:\pi(j)\neq j} S_j = \sum_{j:\pi(j)\neq j} S_{\pi(j)}$. Therefore we have $\sum_{j:\pi(j)\neq j}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) = 2\sum_{j:\pi(j)\neq j} O_j$ and hence $cost_f(O) - cost_f(S) + 2 \cdot cost_s(O) \geq 0$. This proves the desired lemma. ∎

Combining Lemmas 4.1.1 and 4.1.2, we get the following result.

Theorem 4.1.3 *The local search procedure for the metric uncapacitated facility location problem with the neighborhood structure $\mathcal{N}$ given by, $\mathcal{N}(S) = \{S + \{s'\}\} \cup \{S - \{s\} \mid s \in S\} \cup \{S - \{s\} + \{s'\} \mid s \in S\}$ has a locality gap of at most 3.*

The algorithm described above extends to the case when the clients $j \in C$ have arbitrary demands $d_j \geq 0$ to be served. We now show how to use scaling technique from [9] to obtain $1 + \sqrt{2} + \epsilon \approx 2.414 + \epsilon$ approximation to the UFL. The main idea is to exploit the asymmetry in the service and facility cost guarantees.

Note that the Lemmas 4.1.1 and 4.1.2 hold for any solution O and not just the optimal solution. We multiply the facility costs by a suitable factor $\alpha > 0$ and solve the new instance using local search.

Theorem 4.1.4 *The metric uncapacitated facility location problem can be approximated to factor $1 + \sqrt{2} + \epsilon$ using a local search procedure.*

Proof. As before, we denote the facility cost and the service cost of an optimum solution O by $cost_f(O)$ and $cost_s(O)$ respectively. Let $cost'_f(A)$ and $cost'_s(A)$ denote the facility and service costs of a solution A in the scaled instance and let S be a locally optimum solution. Then

$$\begin{aligned}
cost_f(S) + cost_s(S) &= \frac{cost'_f(S)}{\alpha} + cost'_s(S) \\
&\leq \frac{cost'_f(O) + 2cost'_s(O)}{\alpha} + cost'_f(O) + cost'_s(O) \\
&= (1+\alpha)cost_f(O) + \left(1 + \frac{2}{\alpha}\right) cost_s(O).
\end{aligned}$$

The inequality follows from Lemmas 4.1.1 and 4.1.2. Now, by setting $\alpha = \sqrt{2}$, we get $cost(S) \leq (1+\sqrt{2})cost(O)$. Thus local search can be used to obtain a $1 + \sqrt{2} + \epsilon$ approximation. ∎

4.1.4 Tight example

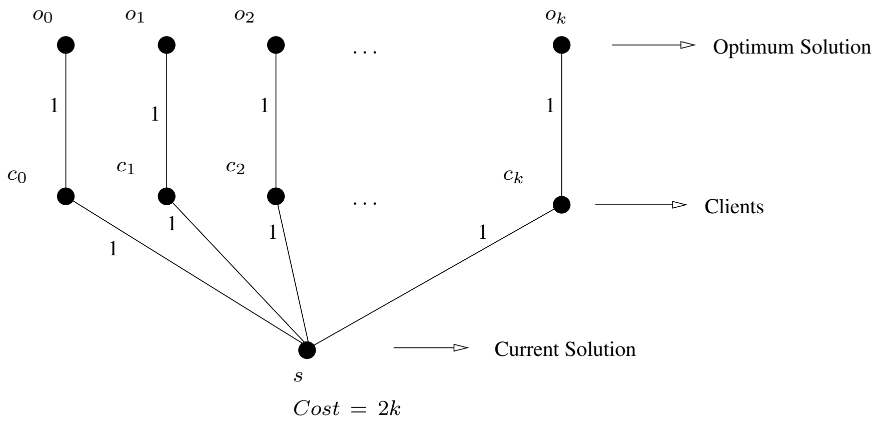


Figure 4.3: Tight example for the locality gap of UFL.

In Figure 4.3, we show an instance where a local optimum has cost at least $3 - o(1)$ times the cost of the global optimum. The locally optimum solution, S, consists of a single facility s while the optimum solution O consists of facilities $\{o_0, o_1, \ldots, o_k\}$. The facility cost of each of the facility in O is zero. The facility cost of s is $2 \cdot k$. The set of clients is $C = \{c_0, c_1, \ldots, c_k\}$. All edges shown have unit lengths and the cost of serving client j by facility f is the length of the shortest path between client j and facility f in the graph. To argue that the solution S is local optimum, we first note that we cannot delete the facility s. It is also easy to verify that we cannot decrease the cost of our solution by either adding any facility from O, or by any swap which involves bringing in a facility from O and deleting s. Thus, S is locally optimum and has cost $3k+1$, while the cost of O is $k+1$. Since the ratio $cost(S)/cost(O)$ tends to 3 as k tends to ∞, our analysis of the local search algorithm is tight.

4.2 The Capacitated Facility Location Problem

In this section, we present a local search algorithm for facility location with soft capacities defined in Section 2.1.4. In the capacitated facility location problem, along with the facility costs $f_i \geq 0$, we are given integer capacities $u_i > 0$ for each $i \in F$. We can open multiple copies of a facility i. Each copy incurs a cost f_i and is capable of serving at most u_i clients. Note that the capacities u_i may be *different* for different facilities i. The problem is to identify a multi-set S of facilities and to serve the clients in C by the facilities in S such that the capacity constraints are satisfied and the sum of the facility costs and the service costs is minimized. Since the clients have unit demands and the facilities have integer capacities, every client will get assigned to a single facility. If a client $j \in C$ is assigned to a facility $\sigma(j) \in S$ then we want to minimize $cost(S) = \sum_{i \in S} f_i + \sum_{j \in C} c_{j\sigma(j)}$. Given an S, the service cost can be minimized by solving a min-cost assignment problem.

Chudak and Shmoys [13] obtained a 3 approximation for this problem when the capacities are uniform by using ideas of filtering the fractional solution of LP relaxation and randomization. The primal-dual algorithm proposed by Jain and Vazirani [27] gives an approximation of 4 which can be further improved to 3.73 by exploiting the asymmetry in guarantees on facility and service cost. Their algorithm works when the capacities are non-uniform as well. We present a local search algorithm for this problem and show a locality gap of 4 thus matching the bound of Jain and Vazirani. The idea of scaling can be applied to our algorithm to improve the guarantee to 3.73. Recently, Mahdian et al. [44] proved an approximation ratio of 2 for this problem by reducing it to a linear-cost facility location problem. We [3] give the first local search algorithm for the capacitated facility location problem with non-uniform capacities. We prove that our local search algorithm has a locality gap of at most 4.

As before, S and O denote the multi-sets of the facilities opened in the locally optimum solution and an optimum solution respectively. The notations used in the analysis of the UFL problem are used here as well.

4.2.1 A local search algorithm

In this section, we prove a locality gap of at most 4 on a local search procedure for the capacitated facility location problem. At each step of the local search, we can either add a single copy of a facility $s' \in F$ or add $l \geq 1$ copies of a facility $s' \in F$ and drop a subset of the open facilities, $T \subseteq S$. For the second operation, l should be such that the clients in $N_S(T)$ can be served by these new copies of s', that is, $l \cdot u_{s'} \geq |N_S(T)|$. So, the neighborhood structure $\mathcal{N}$ is defined by

$$\mathcal{N}(S) = \{S + s' \mid s' \in F\} \cup \{S - T + l \cdot \{s'\} \mid s' \in F, T \subseteq S, l \cdot u_{s'} \geq |N_S(T)|\}. \tag{4.6}$$

where $l \cdot \{s'\}$ represents l new copies of s'. If we service all clients in $N_S(T)$ by the new copies of facility s', the cost of the new solution is at most

$$cost(S) + l \cdot f_{s'} + \sum_{s \in T} \left(-f_s + \sum_{j \in N_S(s)} (c_{js'} - c_{js}) \right).$$

Given a facility $s' \in F$, we use the Procedure `T-Hunt` described in Figure 4.4 to find a subset, $T \subseteq S$, of facilities to close. Here $m = |C|$ is an upper bound on the number of new copies of s' that we need to open. Closing a facility $s \in S$ gives an extra $|N_S(s)|$ clients to be served by the new facility s'. A client $j \in N_S(s)$ now travels an extra distance of at most $(c_{s'j} - c_{sj})$. Thus, closing facility s gives a *saving* of $f_s - \sum_{j \in N_S(s)} (c_{s'j} - c_{sj})$. Due to capacity constraints, a copy of s' can serve at most $u_{s'}$ clients. This motivates us to define the following `Knapsack` problem. For a facility $s \in S$, define $\texttt{weight}(s) = |N_S(s)|$ and $\texttt{profit}(s) = f_s - \sum_{j \in N_S(s)} (c_{s'j} - c_{sj})$. The oracle $\texttt{Knapsack}(W)$ returns a multi-set $T \subseteq S$ such that $\sum_{s \in T} \texttt{weight}(s) \leq W$ and $\texttt{profit}(T) = \sum_{s \in T} \texttt{profit}(s)$ is maximized.

It is interesting to note that the number of choices for a step of local search is exponential in $|S|$ as we allow any subset $T \subseteq S$ to be dropped from the current solution. However, by counting the change in cost due to each such operation in a specific way, we are able to give a polynomial time procedure (the procedure `T-hunt`) to identify a local operation which improves the cost. It might be the case that `T-hunt` is not able to identify a local operation which improves the cost even though such operations exist. However, our analysis will work only with the assumption that `T-hunt` could not find a solution which improves the cost.

```
    Procedure T-Hunt.

1.  For $l = 1$ to $m$ do,
2.    $T \leftarrow Knapsack(l \cdot u_{s'})$.
3.    If $cost(S) + l \cdot f_{s'} -$ profit$(T) < cost(S)$,
        then return $T$.
4.  return "could not find a solution that reduces the cost".
```

Figure 4.4: A procedure to find a subset $T \subseteq S$ of facilities

4.2.2 The analysis

As the output S is locally optimum with respect to additions, Lemma 4.1.1 continues to bound the service cost of S. We restate Lemma 4.1.1 here.

Lemma 4.2.1 (Service cost)

$$cost_s(S) \leq cost_f(O) + cost_s(O).$$

Lemma 4.2.2 *For any $U \subseteq S$ and any $s' \in F$, we have,*

$$\lceil |N_S(U)|/u_{s'} \rceil \cdot f_{s'} + \sum_{s \in U} |N_S(s)| \cdot c_{ss'} \geq \sum_{s \in U} f_s.$$

Proof. The algorithm terminated with the output S. Hence for the solution S and for the facility s', the Procedure `T-Hunt` must have returned "could not find a solution that reduces the cost". Consider the run of the for-loop for $l = \lceil |N_S(U)|/u_{s'} \rceil$. Since $\sum_{s \in U} \texttt{weight}(s) = N_S(U) \leq l \cdot u_{s'}$, the solution T returned by the knapsack oracle has profit at least as large as $\texttt{profit}(U)$. Hence,

$$0 \leq l \cdot f_{s'} - \texttt{profit}(T) \leq l \cdot f_{s'} - \texttt{profit}(U) = l \cdot f_{s'} - \sum_{s \in U} \left(f_s - \sum_{j \in N_S(U)} (c_{js'} - c_{js}) \right).$$

But by triangle inequality we have, $c_{js'} - c_{js} \leq c_{ss'}$. Therefore we have the lemma. ∎

We are now ready to bound the facility cost of S.

Lemma 4.2.3 (Facility cost)

$$cost_f(S) \leq 3 \cdot cost_f(O) + 2 \cdot cost_s(O).$$

To prove the above lemma, we consider a directed graph $G = (V, E)$ with lengths on edges as shown in Figure 4.5, where,

$$V = \{v_s \mid s \in S\} \cup \{w_o \mid o \in O\} \cup \{sink\},$$

$$E = \{(v_s, w_o) \mid s \in S, o \in O\} \cup \ \{(w_o, sink) \mid o \in O\}.$$

The lengths of (v_s, w_o) and $(w_o, sink)$ are c_{so} and f_o/u_o respectively. The cost of routing unit amount of flow along any edge is equal to the length of that edge. We want to simultaneously route $|N_S(s)|$ units of flow from each v_s to the $sink$.

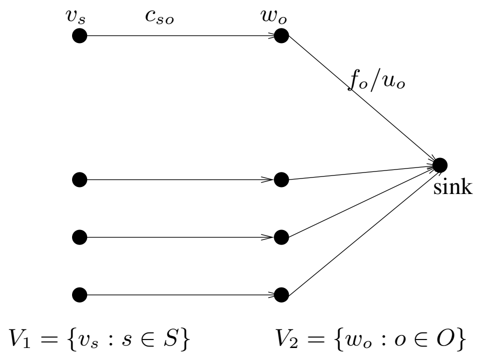


Figure 4.5: The flow graph

Lemma 4.2.4 *We can simultaneously route $|N_S(s)|$ units of flow from each v_s to the $sink$ such that the total routing cost is at most $cost_s(S) + cost_s(O) + cost_f(O)$.*

Proof. Consider a client $j \in C$. If $j \in N_s^o$ then route one unit of flow along the path $v_s \rightarrow w_o \rightarrow sink$. Triangle inequality implies, $c_{so} \leq S_j + O_j$. If for each client we route a unit flow in this manner then the edge $(w_o, sink)$ carries $N_O(o)$ units of flow at cost $|N_O(o)| \cdot f_o/u_o \leq \lceil |N_O(o)|/u_o \rceil \cdot f_o$, which is the contribution of o to $cost_f(O)$. Thus, the routing cost of this flow is at most $cost_s(S) + cost_s(O) + cost_f(O)$. ∎

Since there are no capacities on the edges of graph G, any minimum cost flow must route all $N_S(s)$ units of flow from v_s to the sink, along the shortest path. This would be a path (v_s, w_o, sink), where o is such that $c_{so} + f_o/u_o$ is minimized with ties broken arbitrarily. For each $o \in O$, let $T_o \subseteq S$ denote the set of facilities s that route their flow via w_o in this minimum cost flow. From Lemma 4.2.4, we have,

$$cost_s(S) + cost_s(O) + cost_f(O) \geq \sum_{o \in O} \sum_{s \in T_o} |N_S(s)|(c_{so} + f_o/u_o). \tag{4.7}$$

Now, applying Lemma 4.2.2 to T_o and o, we get,

$$\lceil |N_S(T_o)|/u_o \rceil \cdot f_o + \sum_{s \in T_o} |N_S(s)| \cdot c_{so} \geq \sum_{s \in T_o} f_s.$$

Hence,

$$f_o + |N_S(T_o)|/u_o \cdot f_o + \sum_{s \in T_o} |N_S(s)| \cdot c_{so} \geq \sum_{s \in T_o} f_s.$$

Adding these inequalities for all $o \in O$, we get,

$$\sum_{o \in O} f_o + \sum_{o \in O} \sum_{s \in T_o} |N_S(s)|(c_{so} + f_o/u_o) \geq \sum_{o \in O} \sum_{s \in T_o} f_s = cost_f(S). \tag{4.8}$$

The inequalities (4.7) and (4.8) together imply

$$cost_f(S) \leq 2 \cdot cost_f(O) + cost_s(O) + cost_s(S).$$

This inequality together with Lemma 4.2.1 gives Lemma 4.2.3. Combining Lemmas 4.2.1 and 4.2.3, we obtain the following result.

Theorem 4.2.5 *A local search procedure for the metric capacitated facility location problem where in each step we can either add a facility or delete a subset of facilities and add multiple copies of a facility has a locality gap of at most 4.*

Using an argument similar to the one in Theorem 4.1.4 with $\alpha = \sqrt{3} - 1$ we obtain a $2 + \sqrt{3} + \epsilon \approx 3.732 + \epsilon$ approximation. The tight example given in Section 4.1.4 for the uncapacitated facility location problem shows that a locally optimum solution for this problem can have cost 3 times the cost of the global optimum. However, we do not know of a local minima with a locality gap of 4. So, the question of improving the locality gap or proving that the analysis is tight is open.

Chapter 5

The k-uncapacitated facility location problem

In this chapter, we consider the k-UFL problem defined in Section 2.1.3. We consider the most natural local search algorithm and prove that it has a locality gap of 5. We also establish a connection between the locality gap of the k-UFL problem and the worst-case equilibria of a network service provider game. We first introduce the service provider game and establish its relationship with the locality gap of the k-UFL problem. This is followed by the analysis of the locality gap.

5.1 Motivation

Consider a network in which, for each link, the delay across the link is determined by a function of the traffic on it. Suppose there is a group of agents, each one of whom wants to send a particular amount of flow between a (source, destination) pair. Each agent wants to take the path which is likely to face least delay. This defines a game in which, for each player, there are as many pure strategies as the number of paths between the source and destination. The agent can choose to split his traffic along many different paths. Let the weighted average of the delays across the various paths be the index of the agent's performance. Each agent would like to minimize his average delay. There is also a well defined optimization problem in this setting, which is, what is the best way to route the traffic of all agents so that the average delay of the entire traffic is minimized? A natural question to ask is, what is the worst-case bound on the ratio of the average delay of an equilibrium reached by a set of non-cooperating, selfish agents to the minimum average delay required to route all the traffic? This question, popularly known as, the cost of selfish routing, is an example of a problem in which we seek an informative estimate of the impact of lack of cooperation among a set of selfish agents on the underlying social cost.

Koutsoupias and Papadimitriou [34] gave a mathematical formulation for studying this problem. They formulated the problem in terms of the *Nash equilibrium* attained by a set of independent, non-cooperative agents with rational behavior. In an environment in which each agent is aware of all the alternatives facing all the other agents, Nash equilibrium is a combination of choices (deterministic or randomized), one for each agent, in which, no agent has an incentive to unilaterally move away. The Nash equilibrium is known to deviate from overall optimum in many optimization scenarios. Koutsoupias and Papadimitriou defined *worst case equilibria* as the maximum value that the ratio of the overall optimum to the cost of a Nash Equilibrium can take over the set of all Nash equilibriums. Papadimitriou [47] called it as the *price of anarchy*.

Price of anarchy has been used by Roughgarden and Tardos [52, 50, 51] to analyze games in network design. For example, they showed that the price of anarchy is at most 4/3 when the delay of each link increases linearly with the traffic. For arbitrary, continuous, non-decreasing delay functions, they showed that the price of anarchy can be arbitrarily large. However, they showed that, for a given a traffic between (source, destination) pairs, the worst case delay is no more than the minimum delay when the traffic between each pair is twice the original traffic.

We define a network service provider game that is natural in the context of network design and consider the question of upper bounding the price of anarchy for this game. We first show that the price of anarchy for our game is precisely the locality gap of the k-UFL problem with a specific neighborhood structure. We then show that, the locality gap for the k-UFL with this neighborhood is at most 5. This connection also gives rise to the prospect of upper bounding the price of anarchy of games by considering locality gap of suitably modified optimization problems.

We consider the following service provider game. We are given a network with a distinguished node called the *root*. The remaining nodes are partitioned into two sets: nodes on which clients reside, C, and nodes that can be occupied by service providers, F. The distances between the nodes in the network satisfy the metric property. Assume there are k service providers. Once a subset of them occupy nodes in F, each client is served by the closest service provider. A client, say j, connects to the closest service provider, say i, and makes the VCG payment [56, 14, 20], i.e., the cost of connecting to the second closest service provider. This defines the total revenue accrued by i. On the other hand, the cost incurred by i is the total connecting cost to each of its clients and the connection cost from i to the root node. The difference of the revenue and the cost is the net profit of i.

Each service provider is allowed three kinds of moves:

- Deletion: A service provider who is occupying a node in F may decide to not participate in the game (e.g., if his profit becomes negative).
- Addition: A service provider who is not participating in the network may decide to occupy a vacant node in F.
- Swap: A service provider may move from one node of F to another vacant node.

Note that, at any time in the game, a player either occupies a service location of his choice, or decides not to occupy any location. Informally, the choices of all the players defines a configuration. A configuration of service providers is said to be *Nash equilibrium* if none of the service providers can improve their profit using the three moves stated above. The cost of this Nash equilibrium is the total cost incurred by the service providers. The optimal cost is the minimum cost incurred by at most k service providers in serving all clients. The price of anarchy is supremum, over all Nash equilibria, of the ratio of the cost of a Nash equilibrium to the optimal cost. We formalize these notions in the next section.

The k-UFL problem as described in Section 2.1.3 requires us to open at most k facilities such that the total of the facility cost and the service cost is minimized. The k-UFL problem inherits features of both, the k-median and the uncapacitated facility location problems. We show that, there is a natural correspondence between the service provider game and a natural neighborhood structure for the k-UFL problem. We also show that the price of anarchy of the service provider game is equal to the locality gap of the k-UFL problem with this neighborhood.

Vetta [55] considers similar games arising from maximization versions of the facility location problem and the k-median problem. He gives bounds on the price of anarchy of these games. While the objective function in Vetta's formulation maximizes a social utility function, the objective function in our formulation minimizes a social cost function. The optimization problem in case of maximization turns out to be submodular, and tractable. The optimization problem in our case, the k-UFL is NP-hard.

Jain and Vazirani [27] gave the first approximation algorithm for the k-UFL problem. They showed that their technique for the k-median problem can be modified to work for the k-UFL problem as well. They proved an approximation ratio of 6. This ratio can be improved to 4 by using the techniques of Charikar and Guha [9]. However, these results do not seem to be helpful in analyzing the price of anarchy of the service provider game.

5.2 Preliminaries

5.2.1 Service provider game

In the service provider game, we are given a network with a distinguished node r called *root*. The remaining set of nodes in the network is partitioned into a set of clients C and a set of service locations F. We are also given the distances c_{ij} between $i, j \in F \cup C \cup \{r\}$ that satisfy metric properties. Suppose that there are k service providers. Each service provider is allowed to occupy at most one service location. Two service providers cannot occupy a single service location. The subset of service locations occupied by service providers, say $S \subseteq F$, defines a *configuration*. Note that, not all providers may get assigned to locations. Consider a service location $i \in S$. We use i to denote the location as well as the service provider occupying the location. The context of usage clarifies the reference.

Consider a configuration $S \subseteq F$ with $|S| \leq k$. For a service provider $i \in S$, let $N_S(i)$ be the neighborhood of i, i.e., the set of clients for which i is the closest among the providers in the solution S. To service all the clients in $N_S(i)$ and to connect to the root r, the provider i incurs a total cost of $cost_S(i) = c_{ir} + \sum_{j \in N_S(i)} c_{ij}$. Let the total cost of all the providers in a configuration S be denoted by $cost(S) = \sum_{i \in S} cost_S(i)$. For a client $j \in C$, and a configuration S, let $S_j = \min_{i \in S} c_{ij}$ be the distance of j to the closest provider in S.

Each client connects to the service provider closest to it, but it makes a VCG payment, which is the distance to the second closest service provider. The *revenue* of a service provider $i \in S$ is the total payment it receives from all the clients it serves, i.e., $revenue_S(i) = \sum_{j \in N_S(i)} T_j$ where $T = S - \{i\}$. The profit of a service provider i is $profit_S(i) = revenue_S(i) - cost_S(i)$. Note that both $revenue_S(i)$ and $profit_S(i)$ are with respect to a particular configuration S.

5.2.2 The k-UFL problem

The k-UFL was defined in Section 2.1.3. We are required to open at most k facilities so as to minimize the total of facility cost and service cost. Note that, given a set $S \subseteq F, |S| \leq k$, the assignment of clients to the facilities is straightforward. We assume the cost of an empty solution to be infinity. The strategies allowed for the agents define the following local operations:

- *delete* a facility: if there is $i \in S$ such that $cost(S - i) < cost(S)$, then $S := S - i$.
- *add* a facility: if $|S| < k$ and there is $i \in F \setminus S$ such that $cost(S + i) < cost(S)$, then $S := S + i$.
- *swap* facilities: if there is $i \in S$ and $i' \in F \setminus S$ such that $cost((S - i) + i') < cost(S)$, then $S := (S - i) + i'$.

Formally, the neighborhood structure is given by,

$$\mathcal{N}(S) = \{S + \{s'\} \text{ if } |S| < k\} \cup \{S - \{s\} \mid s \in S\} \cup \{S - \{s\} + \{s'\} \mid s \in S\}. \tag{5.1}$$

We prove the following theorem for the locality gap of the k-UFL problem with this neighborhood (or local operations).

Theorem 5.2.1 *The locality gap of the k-UFL problem with the above neighborhood structure is at most 5.*

5.3 Price of Anarchy and Locality Gap

There is a natural correspondence between an instance of the service provider game and the k-facility location problem. The set of service locations in the game corresponds to the set of facilities in the k-facility location problem. The cost c_{ir} that a provider i incurs in connecting to the root corresponds to the facility cost f_i. The total cost of a configuration S corresponds to the sum of the cost of facilities S in the k-facility location problem and the total service cost.

In this section, we show that a Nash equilibrium in the service provider game corresponds to a local optimum solution in the k-facility location instance with respect to the delete-add-swap local search algorithm.

Lemma 5.3.1 *The profit of a provider s in the configuration S is given by $profit_S(s) = cost(S - s) - cost(S)$.*

Proof. Let $T = S - s$.

$$\begin{aligned}
cost(T) - cost(S) &= \left(\sum_{i\in T} f_i + \sum_{j\in C} T_j\right) - \left(\sum_{i\in S} f_i + \sum_{j\in C} S_j\right) \\
&= \sum_{j\in C}(T_j - S_j) - f_s \\
&= \sum_{j\in N_S(s)} T_j - \left(\sum_{j\in N_S(s)} c_{sj} + f_s\right) \\
&= revenue_S(s) - cost_S(s) \\
&= profit_S(s).
\end{aligned}$$

∎

Theorem 5.3.2 *A configuration $S \subseteq F$ is a Nash equilibrium of an instance of the service provider game if and only if S is a local optimum solution of the corresponding instance of the k-facility location problem with respect to the delete-add-swap local search.*

Proof. Let $S \subseteq F$ be a Nash equilibrium. From the definition, $profit_S(s) = cost(S - s) - cost(S) \geq 0$ for all $s \in S$. Therefore $cost(S)$ cannot be reduced by deleting a facility $s \in S$. Let $S' = S + s$ for some $s \notin S$. Since any provider not in S did not occupy the location s, we have $profit_{S'}(s) \leq 0$. Therefore, from Lemma 5.3.1, we have $cost(S' - s) - cost(S') \leq 0$. Thus $cost(S) \leq cost(S + s)$. Therefore, $cost(S)$ cannot be reduced by adding a facility $s \notin S$. Let $S' = S - s + s'$ for some $s \in S$ and $s' \notin S$. Since the provider s does not move from location s to s', we have $profit_{S'}(s') \leq profit_S(s)$. Let $T = S - s = S' - s'$. We then, have

$$\begin{aligned}
cost(S') - cost(S) &= (cost(T) - cost(S)) - (cost(T) - cost(S')) \\
&= profit_S(s) - profit_{S'}(s') \\
&\geq 0.
\end{aligned}$$

Therefore, $cost(S)$ cannot be reduced by swapping a pair of facilities. Thus, the solution $S \subseteq F$ is indeed a local optimum solution with respect to the delete-add-swap local search.

Similarly, we can show that, if S is a local optimum solution, then it is a Nash equilibrium in the service provider game. ∎

Theorem 5.3.3 *The price of anarchy for the service provider game is at most 5.*

Proof. The proof follows from Theorems 5.2.1 and 5.3.2. ∎

5.4 Locality Gap of the k-UFL Problem

The k-median proof crucially uses the fact that, the global optimum solution has exactly k facilities. However, for the instance of the k-UFL problem derived from the network service provider game, the global optimum may have significantly smaller number of facilities. The analysis of the local search for the UFL problem considers add operations irrespective of the number of facilities in the current solutions. If the k-facility location solution has exactly k facilities, we cannot add a facility to reduce its cost. It is for these reasons that, the analyses of k-median problem, and the UFL problem cannot be extended trivially to help us obtain a locality gap for the k-facility location problem.

Let S denote a local optimum solution to the k-facility location problem and let O denote a global optimum solution. If $|S| < k$, then S is a local optimum with respect to the addition operation as well. So the analysis for the UFL is directly applicable and we have the following theorem.

Lemma 5.4.1 *If $|S| < k$, then $cost(S) \leq 3 \cdot cost(O)$.*

Therefore, in the rest of the analysis we assume that $|S| = k$. This implies that we have to prove our bound on the locality gap using only swap and delete operations.

5.4.1 Notations

We use the notations introduced in the analysis of the k-median problem, and the UFL problem. The concept of neighborhood of a facility, notion of capture are also used. We need further classification of bad facilities. A facility in S is called "1-bad", if it captures exactly one facility in O. It is called "2+bad" if it captures at least 2 facilities in O.

We now define a 1-1 and onto function $\pi : N_O(o) \rightarrow N_O(o)$ for each $o \in O$. We need to modify the construction for π used in the analysis of the k-median problem. This modification guarantees an extra property which is used crucially in the analysis of the k-facility location problem. We first consider a facility $o \in O$ that is not captured by any facility in S. Let $M = |N_O(o)|$. Order the clients in $N_O(o)$ as $c_0, \ldots, c_{M-1}$ such that for every $s \in S$, the clients in N_s^o are consecutive, that is, there exists p, q, $0 \le p \le q \le M$ such that $N_s^o = \{c_p, \ldots, c_{q-1}\}$. Now, define $\pi(c_i) = c_j$ where $j = (i + \lfloor M/2 \rfloor)$ modulo M.

Lemma 5.4.2 *For each $s \in S$, we have $\pi(N_s^o) \cap N_s^o = \emptyset$.*

The proof for the above lemma is essentially the same as the one presented in Section 3.3.1.

Now, we consider a facility $o \in O$ that is captured by a facility $s \in S$. Note that $|N_s^o| > |N_O(o)|/2$. Therefore, $0 < 2|N_s^o| - |N_O(o)| \le |N_s^o|$. Consider an arbitrary subset $N \subseteq N_s^o$ of size $2|N_s^o| - |N_O(o)|$. For each $j \in N$, we define $\pi(j) = j$. Note that $|N_O(o) \setminus N_s^o| = |N_s^o \setminus N|$. We pair each client $j \in N_O(o) \setminus N_s^o$ with a unique client $j' \in N_s^o \setminus N$ and define $\pi(j) = j'$ and $\pi(j') = j$.

Note that the function $\pi : N_O(o) \to N_O(o)$ for each $o \in O$ as defined in the two cases above satisfies the following properties.

P1. If $s \in S$ does not capture $o \in O$, then $\pi(N_s^o) \cap N_s^o = \emptyset$.

P2. If $s \in S$ captures $o \in O$ and if $j \in N_s^o$ is such that $\pi(j) \in N_s^o$, then $\pi(j) = j$.

P3. We have $\{j \in N_O(o) \mid \pi(j) \neq j\} = \{\pi(j) \in N_O(o) \mid \pi(j) \neq j\}$ for each $o \in O$.

5.4.2 Deletes and Swaps considered

Since S is a local optimum solution, its cost cannot be reduced by doing any deletions and swaps. Since $|S| = k$, we cannot add a facility to reduce the cost. For any $s \in S$, we have $cost(S - s) \ge cost(S)$. For any $s \in S$ and $o \in O$, we have $cost(S - s + o) \ge cost(S)$. We now carefully consider some delete and swap operations. If we delete $s \in S$, we reroute the clients in $N_S(s)$ to other facilities in $S - s$. If we swap $s \in S$ and $o \in O$, we reroute the clients in $N_S(s)$ and a subset of clients in $N_O(o)$ to the facilities in $S - s + o$. The assignment of the other clients is not changed. For each of the operations considered, we obtain an upper bound on $cost(S') - cost(S) \ge 0$, where S' is the solution obtained after the operation. We then add these inequalities to prove Theorem 5.2.1.

We consider the following operations.

1. Each 1-bad facility $s \in S$ is swapped with the facility $o \in O$ that it captures. The clients $j \in N_O(o)$ are rerouted to o. The clients $j \in N_S(s) \setminus N_O(o)$ are rerouted to $s' \in S$ that serves $\pi(j)$ in S. Since s does not capture any $o' \neq o$, P1 implies that $s' \neq s$ and the rerouting is feasible. We call these swaps, **Type 1** operations.

2. Each 2+bad facility $s \in S$ is swapped with the nearest facility $o \in O$ that it captures. All the clients in $N_O(o)$ are rerouted to o. Consider a facility $o' \neq o$ captured by s. Such a facility o' is called a *far* facility. The clients $j \in N_s^{o'}$ such that $\pi(j) = j$ are rerouted to o. The remaining clients $j \in N_S(s)$ are rerouted to $s' \in S$ that serves $\pi(j)$ in S. From P1 and P2, such rerouting is feasible. We call these swaps, **Type 2** operations.

3. Let $G \subseteq S$ be the subset of facilities in S that are not considered in Type 1, or Type 2 operations. Note that G is precisely the set of good facilities. Let $R \subseteq O$ be the subset of facilities in O that are not considered in Type 1 or Type 2 operations. These are the facilities in O that are not captured by any facility in S or are far facilities. Since $|S| = k \geq |O|$ and $|S \setminus G| = |O \setminus R|$, we have $|G| \geq |R|$. Let $G = \{s_1, \ldots, s_{|G|}\}$ and $R = \{o_1, \ldots, o_{|R|}\}$. We swap s_i with o_i for $1 \leq i \leq |R|$. For each of these swaps, we reroute the clients in $N_S(s_i) \cup N_O(o_i)$ as follows. All the clients in $N_O(o_i)$ are rerouted to o_i. The clients $j \in N_S(s_i) \setminus N_O(o_i)$ are rerouted to $s' \in S$ that serves $\pi(j)$ in S. Since s_i is a good facility, P1 implies $s' \neq s_i$ and hence this rerouting is feasible.

   We consider $|G| - |R|$ more operations as follows. For each i such that $|R| + 1 \leq i \leq |G|$, we delete s_i. After such a deletion, we reroute the clients $j \in N_S(s_i)$ to $s' \in S$ that serves $\pi(j)$ in S. Again, Property P1 ensures that $s' \neq s$ and hence this rerouting is feasible.

   We call these $|R|$ swaps and $|G| - |R|$ deletions, **Type 3** operations.

4. Let $R' \subseteq O$ be the subset of far facilities in O. Since no far facility is considered in Type 1, or Type 2 operations, we have $R' \subseteq R$. Let $R' = \{o_1, \ldots, o_{|R'|}\}$. Recall that $G = \{s_1, \ldots, s_{|G|}\}$ is the set of good facilities. We consider $|R'|$ swaps as follows. For each i such that $1 \leq i \leq |R'|$, we swap s_i with o_i. The clients $j \in N_O(o_i)$ such that $\pi(j) = j$ are rerouted to o_i. The clients $j \in N_S(s_i)$ are rerouted to $s' \in S$ that serves $\pi(j)$ in S. The remaining clients are not rerouted. We call these $|R'|$ swaps, **Type 4** operations.

Since S is a local optimum solution, the increase in the facility and service costs after each operation considered above is at least zero. In the sections to follow, we bound this increase due to all the 4 types of operations together.

5.4.3 Bounding the increase in the facility cost

Let $fac(S)$ and $fac(O)$ denote $\sum_{i \in S} f_i$ and $\sum_{i \in O} f_i$ respectively. In Type 1, 2, and 3 operations, each facility in O is brought in exactly once and each facility in S is taken out exactly once. Thus, in these operations the increase in the facility cost is exactly $fac(O) - fac(S)$. In Type 4 operations, each far facility is brought in exactly once and some good facilities are taken out exactly once. Thus, the increase in facility cost in these operations is at most $\sum_{o:\text{far}} f_o \leq fac(O)$. The overall increase in the facility cost due to all the 4 types of operations is at most

$$2 \cdot fac(O) - fac(S). \tag{5.2}$$

5.4.4 Bounding the increase in the service cost

We partition the set of clients C into three categories, "white", "gray", and "black" as follows. We call a facility $o \in O$ a *near* facility if it is captured by some $s \in S$ and o is not a far facility.

1. A client $j \in C$ is called "white" if $\pi(j) = j$ and j is served by a near facility in O.
2. A client $j \in C$ is called "gray" if $\pi(j) = j$ and j is served by a far facility in O.
3. A client $j \in C$ is called "black" if $\pi(j) \neq j$.

Recall that a client $j \in N_s^o$ for $s \in S$ and $o \in O$ is rerouted only when, either o is brought in, and/or s is taken out.

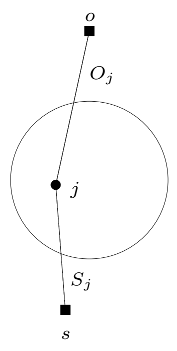


Figure 5.1: Rerouting a client $j \in N_O(o)$ when o is brought in

Lemma 5.4.3 *The total increase in the service cost of a white client $j \in C$ in all the 4 types of operations is at most $O_j - S_j$.*

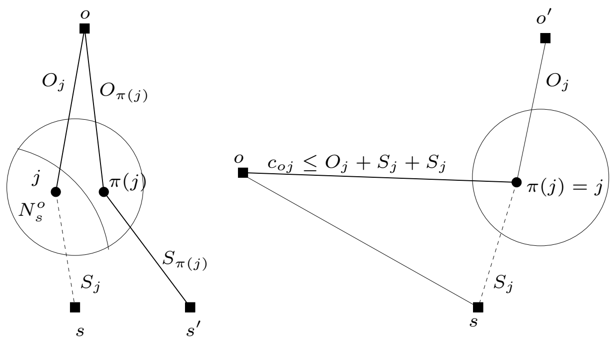


Figure 5.2: Rerouting a client $j \in N_S(s)$ when s is taken out, and the facility serving it in the optimal solution is not brought in: Right hand side shows the rerouting when $\pi(j) \notin N_S(s)$. Left hand side shows the rerouting when $\pi(j) = j$ and s is a 2+bad facility. We swap s with o, its near facility. Here, $j \in N_O(o')$ and s captures o'.

Proof. Let $j \in N_s^o$ where $s \in S$ is a 1-bad or 2+bad facility that captures the near facility $o \in O$. Note that we swap s with o once - either as a Type 1 operation, or as a Type 2 operation. The increase in the service cost of j in this swap is $O_j - S_j$ (Figure 5.1). Since s or o are not considered in Type 3 or 4 operations, the client j is not rerouted in these operations. ∎

Lemma 5.4.4 *The total increase in the service cost of a gray client $j \in C$ in all the 4 types of operations is at most $3O_j - S_j$.*

Proof. Let j be served by a far facility o' in O. Let s be the 2+bad facility in S that captures o'. Since $\pi(j) = j$, from P1, we have $j \in N_s^{o'}$. Let o be the closest facility in O that s captures. Note that $c_{so} \leq c_{so'}$. A gray client is not rerouted in Type 1 operations. In Type 2 operations, j is rerouted to o (RHS Figure 5.2). The increase in the service cost of j is at most $c_{jo} - c_{js}$. Since $c_{jo} \leq c_{js} + c_{so} \leq c_{js} + c_{so'} \leq c_{js} + c_{js} + s_{jo'}$, the increase is at most $c_{js} + c_{jo'} = S_j + O_j$. In Type 3 and 4 operations, the increase in the service cost of j is $O_j - S_j$ each (Figure 5.1). Thus, the total increase in the service cost of j due to all the 4 types of operations is at most $(O_j + S_j) + 2(O_j - S_j) = 3O_j - S_j$. ∎

Lemma 5.4.5 *The total increase in the service cost of a black client $j \in C$ in all the 4 types of operations is at most $2D_j + O_j - S_j$ where $D_j = O_j + O_{\pi(j)} + S_{\pi(j)} - S_j$.*

Proof. Let $j \in N_s^o$ for facilities $s \in S$ and $o \in O$. We first bound the increase in the service cost of a black client j due to operations of Type 1, 2, and 3. Amongst these operations, there is exactly one swap in which o is brought in. This swap contributes $O_j - S_j$ to the increase in service cost of j. The facility s may be either deleted once or considered in a swap with $o' \in O$. If it is considered in a swap with $o' \in O$ such that $o' = o$, then the increase in service cost of client j has already been accounted for in $O_j - S_j$. If s is deleted or considered in a swap with $o' \in O$ such that $o' \neq o$, j is rerouted to s' that serves $\pi(j)$ in S and the increase in its service cost is at most $c_{js'} - c_{js} \leq c_{jo} + c_{o\pi(j)} + c_{\pi(j)s'} - c_{js} = O_j + O_{\pi(j)} + S_{\pi(j)} - S_j = D_j$ (LHS Figure 5.2). Thus, the total increase in service cost of j due to Type 1,2, and 3 operations is at most $D_j + O_j - S_j$.

In Type 4 operations, since $\pi(j) \neq j$, the client j is rerouted only when s is (possibly) taken out. In such a case, it is rerouted to s' that serves $\pi(j)$ in S (LHS Figure 5.2) and the increase in its service cost is again at most $c_{js'} - c_{js} \leq c_{jo} + c_{o\pi(j)} + c_{\pi(j)s'} - c_{js} = O_j + O_{\pi(j)} + S_{\pi(j)} - S_j = D_j$. Thus the total increase in the service cost of j in all 4 types of operations is at most $2D_j + O_j - S_j$. ∎

Let $serv(S)$ and $serv(O)$ denote $\sum_{j\in C} S_j$ and $\sum_{j \in C} O_j$ respectively.

Lemma 5.4.6 *The total increase in the service cost of all the clients in all the 4 types of operations is at most* $5 \cdot serv(O) - serv(S)$.

Proof. Lemmas 5.4.3,5.4.4, and 5.4.5, imply that the the total increase in the service cost of all the clients in all the 4 types of operations is at most

$$\sum_{j:\text{white}} (O_j - S_j) + \sum_{j:\text{gray}} (3O_j - S_j) + \sum_{j:\text{black}} (2D_j + O_j - S_j).$$

Now from property P3 of the function π, we have $\sum_{j:\text{black}} S_j = \sum_{j:\text{black}} S_{\pi(j)}$ and $\sum_{j:\text{black}} O_j = \sum_{j:\text{black}} O_{\pi(j)}$. Hence,

$$\begin{aligned} \sum_{j:\text{black}} D_j &= \sum_{j:\text{black}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \\ &= 2 \sum_{j:\text{black}} O_j. \end{aligned}$$

Therefore, the total increase in the service cost of all the clients is at most $\sum_{j \in C}(5 \cdot O_j - S_j) = 5 \cdot serv(O) - serv(S)$. ∎

5.4.5 Bounding the increase in the total cost

From Equation (5.2) and Lemma (5.4.6), we get that the sum of the increases in the total cost over all operations is at most $2 \cdot fac(O) - fac(S) + 5 \cdot serv(O) - serv(S)$. Since S is a local optimum, in each operation the total cost increases by a non-negative amount. Hence, $2 \cdot fac(O) - fac(S) + 5 \cdot serv(O) - serv(S) \geq 0$, which implies that $cost(S) = fac(S) + serv(S) \leq 2 \cdot fac(O) + 5 \cdot serv(O) \leq 5 \cdot cost(O)$. This proves Theorem 5.2.1.

The k-median problem is a special case of the k-facility location problem in which all facility costs are zero. As we have already seen, for the k-median problem, there is a family of instances in which the ratio of the costs of a local optimum and a global optimum is arbitrarily close to 5. Thus, our analysis of the locality gap for the k-facility location problem is tight. However, the special case in which the facility costs are distances to a fixed node may have lower locality gap.

Chapter 6

The universal facility location problem

In this chapter, we consider the Universal Facility Location Problem (UniFL) defined in Section 2.1.5. In this problem, the cost of a facility is given by a continuous, non-decreasing function of the amount of demand that it serves. Capacitated facility location problems are special cases of the UniFL problem. Recently, Pál and Mahdian [42] gave a local search heuristic with a locality gap of 8 for this problem. We generalize one of the operations considered by them and show an improved locality gap of 7. Zhang et al. [58] also obtained the same locality gap by modifying the analysis of Pál and Mahdian. We thank them for communicating their result to us and allowing us to include this result here.

6.1 Preliminaries

The UFL problem formulation assumes that the cost of a facility is independent of the number of clients that it serves. Different variations of capacitated facility location are formulated to capture the fact that a facility could have capacity constraints and the fact that the cost of a facility depends on the amount of demand that it serves. The ∞-CFL problem assumes that the cost of a facility increases linearly in the amount of demand that it serves modulo its capacity. While this is an improvement over the UFL formulation, it ignores the situations in which the capacity constraint at a particular location is not allowed to be violated. Consider building of bridges across a river formulated as facility location. The building technology may place a limit on the number of vehicles that cross the bridge in a single day. Similarly, if placing of replicated servers is formulated as facility location, the number of clients simultaneously served by a server is also bounded. The 1-

CFL problem discussed in Section 2.1.5 allows us to specify hard capacities. The UniFL problem generalizes the ∞-CFL and 1-CFL problem formulations by allowing the cost of a facility to be any non-decreasing, continuous function of the amount of demand that it serves.

It remains an open problem to obtain useful lower bounds for the 1-CFL and the UniFL instances using linear programs. Of all the techniques discussed in Section 2.2, only local search yields any non-trivial approximation to these problems. Pál et al. [46] first gave a local search heuristic with a locality gap of 9 for the 1-CFL problem with non-uniform capacities. For the ∞-CFL problem, we introduced the operation in which multiple copies of a facility were opened and multiple facilities were closed. Generalizing the idea, [46] introduced two operations, *open_one_close_many*, and *close_one_open_many*. In the open_one_close_many operation, multiple facilities are closed and a facility whose capacity is equal to or greater than the combined capacity of the closed facilities is opened. The close_one_open_many is defined analogously. The rerouting is defined as: the clients served by the closed facility (facilities) are served by the opened facilities (facility). Subsequently, Pál and Mahdian [42] showed that the UniFL has a locality gap of 8 by using even more powerful local operations. They introduced an operation of growing the capacities of a subset of facilities and shrinking the capacities of a subset of facilities. The demand of the shrinking facilities is rerouted to the growing facilities via a pivot facility. Recently, Zhang et al. [59] obtained a locality gap of 6 for the 1-CFL problem by further improving the analysis of Pál and Mahdian. We generalize one of the operations used by Pál and Mahdian to improve the locality gap of the UniFL problem to 7.

The UniFL problem is formulated as the following linear program:

$$\begin{array}{rcll}
\text{minimize} & \sum_{i\in F} G_i(u_i) & + & \sum_{i\in F, j\in C} c_{ij}x_{ij} \\
\text{subject to:} & & & \\
\sum_{i\in F} x_{ij} & = & d_j & \forall j\in C \\
\sum_{j\in C} x_{ij} & \leq & u_i & \forall i\in F \\
u_i, x_{ij} & \geq & 0 & \forall i\in F, \forall j\in C
\end{array}$$

The demand of a client $j \in C$ is denoted by d_j. Let $D = \sum_{j\in C} d_j$. The capacity installed at a facility $i \in F$ is denoted by u_i. For each facility $i \in F$, the facility cost is given by $G_i()$, which is a non-decreasing, continuous function of the capacity installed at i. The assignment vector x is two dimensional and u is the capacity vector. Specifically, the capacity at a facility $i \in F$ is given by u_i and the amount of demand of $j \in C$ served by $i \in F$ is given by x_{ij}.

6.2 Local Search Heuristic

Pál and Mahdian [42] used two local operations to prove a locality gap of 8 for the UniFL problem. The first operation allowed the capacity of a facility to be increased. The second operation allowed to increase and decrease the capacity of multiple facilities at the same time. In this operation, the demand served by the facilities whose capacity decreases is rerouted to the facilities whose capacity increases through a pivot facility. This operation is called a pivot operation. We show that by generalizing this operation to allow rerouting through two pivots, we can improve the locality gap to 7.

The local search heuristic starts with any feasible solution to the UniFL problem instance, and at each step, considers the following operations to improve the cost of the solution:

- **add**(s, δ) **:** Increase the allocated capacity at facility s by δ. For this operation, we compute the actual change in cost, i.e., $G_s(u_s + \delta) - G_s(u_s) + C_s(S') - C_s(S)$ where u_s is current allocated capacity of s. $C_s(S)$, and $C_s(S')$ are the service cost before, and after the operation. As the number of choices for s, and δ are polynomially bounded in input size, add operation can be implemented efficiently. Here, we assume that the demands of the clients are not very large numbers. The costs $C_s(S)$ and $C_s(S')$ are computed by solving minimum cost network flow problems.

- **Single_Pivot**(s, Δ) **:** This operation allows us to implement a limited version of increasing the allocated capacities at multiple facilities, and decreasing the allocated capacities at multiple facilities. Specifically, the vector Δ indicates the change in allocated capacity at each facility. A facility i is said to *shrink* if $\Delta_i < 0$, and it is said to *grow* if $\Delta_i > 0$. Each shrinking facility i sends $|\Delta_i|$ amount of its demand to the pivot s. For each growing facility i, Δ_i amount of demand is then routed from s. Note that, a valid Single_Pivot operation satisfies the property that $\sum_{i \in F} \Delta_i = 0$. We compute the estimated cost of this operation as $\sum_{i \in F}(G_i(u_i + \Delta_i) - G_i(u_i) + c_{si}|\Delta_i|)$. Note that, this is only an upper bound on the change in cost after the operation. The analysis depends on the fact that, at local minima, there is no operation with a negative upper bound.

Pál and Mahdian used the above two operations to prove a locality gap of 8. We need the following modification to the pivot operation to help us prove a locality gap of 7.

- **Double_Pivot**$(s_1, s_2, \Delta^1, \Delta^2)$ **:** This operation is a generalization of the previous operation to redistribute demand around two pivots. Δ^1 specifies the rerouting of demand through s_1, and Δ^2 specifies the rerouting of demand through s_2. For a valid operation, $\sum_{i \in F} \Delta_i^1 = 0$, and $\sum_{i \in F} \Delta_i^2 = 0$. The cost of this operation is estimated in a manner similar to previous operation, i.e., $\sum_{i \in F}(G_i(u_i + \Delta_i^1 + \Delta_i^2) - G_i(u_i) + c_{s_1 i}|\Delta_i^1| + c_{s_2 i}|\Delta_i^2|)$.

6.3 Implementation

Consider a solution S given by (u, x). We first show how to identify if there exists a Single_Pivot operation which improves the cost. For a given facility $s \in F$, we compute a Δ for which the estimated cost of Single_Pivot(s, Δ) is minimum. If the estimated cost is negative, then it represents an operation which improves the cost. By iterating over all the facilities in F, we can check if there exists a Single_Pivot operation which improves the cost.

In the Single_Pivot operation, the rerouting cost is calculated by first collecting the demand of the shrinking facilities at the pivot and then redistributing the collected demand to the growing facilities. If δ is the change in the capacity of a facility $i \in F$ and s is the pivot, then the change in cost is given by $G_i(u_i + \delta) - G_i(u_i) + c_{si} \cdot |\delta|$. We formulate a dynamic program based on this observation to identify the least cost operation pivoted at s.

For each facility $i \in F$, we consider the function $g_i(\delta) = c_{si}.|\delta| + G_i(u_i + \delta) - G_i(u_i)$ for $\delta \in R_i = \{-u_i, \ldots, D - u_i\}$. Its value is made arbitrarily large for other values of δ. Essentially, $g_i(\delta)$ gives the change in cost if $\Delta_i = \delta$ (irrespective of whether it shrinks or grows). The range R_i is chosen to ensure that only feasible operations are allowed. We then compute a two dimensional table a which has $|F|$ rows and $2D + 1$ columns indexed from $-D$ to D. The entry $a[i, w]$ contains the minimum cost of collecting an excess (when $w > 0$) or a deficiency (when $w < 0$) of w units at s by modifying capacities of facilities from 1 to i. This table can be built dynamically as follows:

$$a[i, w] = \begin{cases} g_1(-w) & \text{if } i = 1 \\ min_{\delta \in R_i}\{g_i(\delta) + a[i-1, w+\delta]\} & \text{otherwise} \end{cases}$$

The least cost of a valid pivot operation pivoted at s is given by $a[n, 0]$. If this quantity is negative then there exists an operation pivoted at s which is beneficial. By iterating on all the facilities in F, we can determine if there exists a beneficial Single_Pivot operation.

The above ideas can be used to implement Double_Pivot operation as well. We need to compute two vectors Δ^1 and Δ^2 which are pivoted through s_1, s_2. The details of the implementation remain same except that the function g_i now takes two arguments, δ_1, and δ_2 corresponding to the two pivot operations. In particular,

$$g_i(\delta_1, \delta_2) = \sum_{l=1,2} c_{s_l i}|\delta_l| + G_i(u_i + \delta_1 + \delta_2) - G_i(u_i)$$

We now need to build a three dimensional table in a dynamic fashion as follows:

$$a[i, w_1, w_2] = \begin{cases} g_1[-w_1, -w_2] & \text{if } i = 1 \\ min_{\delta_1,\delta_2}\{g_i[\delta_1, \delta_2] + a[i-1, w_1 + \delta_1, w_2 + \delta_2]\} & \text{otherwise} \end{cases}$$

The range of the function g_i's are restricted by $\delta_1 + \delta_2 \in R_i$. Both the operations have been implemented assuming that the demands are small integers and the cost functions are step functions at integer intervals. This discretization assumption can be removed and $(1+\epsilon)$ approximate versions of these operations implemented by constructing dynamic programs over costs instead of capacities. This has already been demonstrated by Pál and Mahdian in [42].

6.4 Locality Gap

As always, we use S to denote a local optimum solution and S^* to denote the optimal solution. For a facility $i \in F$, u_i denotes the capacity installed by S and u_i^* denotes the capacity installed by S^*. We bound the service cost of S using ideas similar to those used for the UFL problem and the ∞-CFL problem. The following lemma holds.

Lemma 6.4.1 *The service cost of S, $C_s(S)$ is bounded by the cost of the optimal solution, i.e, $C_s(S) \le C_f(S^*) + C_s(S^*)$.*

Proof. If the cost of the solution S can be improved by doing add(s_1, δ_1), and add(s_2, δ_2) simultaneously, then one of the operations must improve the cost of the solution by itself. This property can be generalized. If there exists a set of add operations ADD, which improves the cost of the solution, then there must exist an add operation add$(s_i, \delta_i) \in$ ADD which improves the cost of the solution. As S is a local minima, there is no add operation which improves the cost. So, any set of add operations cannot improve the cost of S. Let us consider a set of add operations in which the capacity of each facility $i \in F$ is set to $\max(u_i, u_i^*)$. The change in the facility cost due to these operations is bounded by $C_f(S^*)$. Also, the capacities at the facilities is sufficient to achieve the service cost of S^*. Therefore, the change in cost due to the above operations is upper bounded by $C_s(S^*) + C_f(S^*) - C_s(S)$. As S is a local minima, $C_s(S) \le C_f(S^*) + C_s(S^*)$. ∎

Bounding the facility cost is harder than bounding the service cost as seen in the UFL problem and the ∞-CFL problem. We use the following scheme to bound the facility cost of S. The main idea is to set up an instance of transshipment problem such that, for each facility $i \in F$, the new capacity is equal to its capacity in S^*. If a demand x of $s \in S$ is assigned to $t \in S^*$, then the cost of reassignment is assumed to be $x{\cdot}c_{st}$. We first show that there is a minimum cost transshipment which has a desirable structure. We then use the structure to define a set of Single_Pivot and Double_Pivot operations on S to bound $C_f(S)$.

6.4.1 A transshipment problem

For a facility $i \in F$, $u_i = \sum_{j \in C} x_{ij}$, and $u_i^* = \sum_{j \in C} x_{ij}^*$, i.e, the capacities at each facility is equal to the demand it serves in the solution. Let $\rho_i = u_i - u_i^*$. If $\rho_i > 0$ then, it denotes the excess at node i and if $\rho_i < 0$ then, it denotes the deficiency at facility i. As both S and S^* serve the same amount of demand, $\sum_{i \in F} \rho_i = 0$. We set up a transshipment problem between the facilities in $F_e = \{i \in F | \rho_i > 0\}$ (called *sources*) and the facilities in $F_d = \{i \in F | \rho_i < 0\}$ (called *sinks*). For each facility $s \in F_e$, we want to transfer ρ_s of its demand to facilities in F_d. Each facility $t \in F_d$ can absorb atmost ρ_t additional demand. If a demand of $\rho_{st} \leq \rho_s$ is routed from $s \in F_e$ to $t \in F_d$ then, the cost of this routing is $\rho_{st} c_{st}$. The goal of the transshipment problem is to find a minimum cost flow satisfying the above constraints. The transshipment problem is given by the following linear program.

$$\begin{array}{rrcll}
\text{minimize} & \sum_{s \in F_e, t \in F_d} c_{st} y_{s,t} & & & \\
\text{subject to:} & & & & \\
\sum_{t \in F_d} y_{s,t} & & = & \rho_s & \forall s \in F_e \\
\sum_{s \in F_e} y_{s,t} & & \leq & |\rho_t| & \forall t \in F_d \\
y_{s,t} & & \geq & 0 & \forall s \in F_e, \forall t \in F_d
\end{array}$$

The solution to the transshipment problem can be thought of as an imaginary **swap** operation. A swap operation is essentially a Single_Pivot operation without the pivot, i.e, the demand from shrinking facilities is directly sent to a subset of growing facilities. The swap operation is very powerful. If it can be implemented efficiently, the analysis of the local search for UFL can be extended to prove a locality gap of 3 for the UniFL problem. However, we do not know of an efficient way to implement a swap operation. Instead, we use the structure of the transshipment problem defined above to specify a set of Single_Pivot and Double_Pivot operations which help us to prove a locality gap of 7. This is achieved in two steps:

- We show that the minimum cost of transshipment is bounded by the sum of service costs of S and S^*.
- We then show that the minimum cost transshipment has a desired forest structure. Furthermore, this structure can be used to define an appropriate set of operations.

Lemma 6.4.2 *The transshipment problem has a solution of cost at most* $C_s(S) + C_s(S^*)$.

Proof. Let us consider the assignment vectors x, and x^* of our solution and optimal solution respectively. We interpret the assignment vectors as sending x_{ij} amount of flow from i to j. We can extend this notion to compare two assignment vectors by considering the difference in their flow values. So, for a client j and facility i, let $fl_{ij} = x_{ij} - x^*_{ij}$. If $fl_{ij} > 0$, then there is a flow of fl_{ij} from i to j. If $fl_{ij} < 0$, then there is a flow of $|fl_{ij}|$ from j to i. It is easy to see that the amount of flow directed into a client is equal to the amount of flow directed out of it. For a facility $s \in F_e$, the flow going out of s is more than the flow coming into s. This difference is equal to its excess. Similarly, for a facility $t \in F_d$, the amount of flow going out of t is less than the amount of flow coming into t and this difference is equal to its deficiency. The network of flows defined above satisfies all the constraints imposed by the transshipment problem. The cost of these flows is at most $\sum_{i \in F, j \in C} x_{ij}c_{ij} + \sum_{i \in F, j \in C} x^*_{ij}c_{ij} = c_s(S) + c_s(S^*)$. ∎

6.4.2 Structure of Minimum Transshipment

A solution y to the transshipment problem can be thought of as a graph with $F_e \cup F_d$ being the set of nodes and the set of non-zero y_{st}'s defining the set of edges. We claim that:

Lemma 6.4.3 *There exists an optimal solution to the transshipment problem which does not contain cycles.*

Proof. Let y denote a solution to the transshipment problem. Let H be a cycle in the graph as defined above. Note that, by definition of the transshipment problem, H is an even cycle of length at least 4 with alternate nodes belonging to F_e and F_d respectively. It is easy to verify that, either the cost of the transshipment can be reduced by augmenting flows along shorter edges and attenuating the flow along longer edges while satisfying all constraints of transshipment or, the cycle can be broken without increasing the cost. This argument can be applied iteratively to transform y to an optimal solution without any cycles. ∎

So an optimal solution to the transshipment problem can be thought of as a collection of bipartite trees edges leading between alternate layers of F_e and F_d. Let $\mathcal{F}$ denote the collection of these trees. We root each tree at an arbitrary facility in F_d. Consider a tree $\mathcal{T}$ rooted at a node $r \in F_d$. For a node $i \in \mathcal{T}$, $K(i)$ denotes the set of children of i in $\mathcal{T}$. So, $K(t) \subseteq F_e$ if $t \in F_d$ and $K(s) \subseteq F_d$ if $s \in F_e$. For a node $t \in F_d$ which is in $\mathcal{T}$ and is not a leaf node, $\mathcal{T}_t$ denotes the subtree rooted at t of depth at most two (See Fig. 6.1). For each such $\mathcal{T}_t$ over all the trees in the forest, we define a set of Single_Pivot, and Double_Pivot operations. The set of the operations considered by us satisfy the following properties:

P1. For each facility $s \in F_e$, the capacity u_s is reduced to u_s^* exactly once. Such an operation is said to *close* the facility $s \in F_e$.

P2. For each facility $t \in F_d$, the capacity of u_t is increased to at most u_t^* atmost three times. Such an operation is said to *open* the facility $t \in F_d$.

P3. For an edge e in the forest, the total amount of flow due to rerouting of all the operations is at most three times the flow on e in the optimal transshipment.

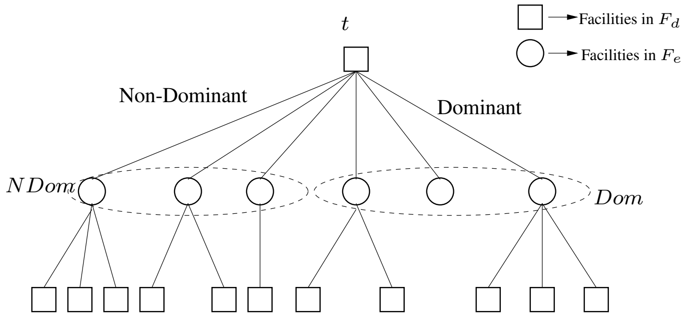


Figure 6.1: The subtree $\mathcal{T}_t$

6.4.3 Analysis

Let y denote the assignment of the optimal transshipment with the tree property as shown in previous section. Consider a subtree $\mathcal{T}_t$ rooted at a facility $t \in F_d$. We classify the facilities in $K(t) \subseteq F_e$ into two categories. A facility $s \in K(t)$ is called *dominant* if at least half of its demand is served by t (i.e. $y_{s,t} \geq \sum_{t' \in F_d} y_{s,t'}$). A facility $s \in K(t)$ is called *non-dominant* if it is not dominant. We denote the set of dominant facilities by Dom, and the set of non-dominant facilities by $NDom$(See Fig. 6.1). Further, we order the facilities in $NDom$ by the non-decreasing order of the demand served by t. If there are k facilities in $NDom$ they are ordered $s_1, s_2, \ldots, s_k$ such that $y_{s_1,t} \leq y_{s_2,t} \leq \ldots \leq y_{s_k,t}$(See Fig. 6.2).

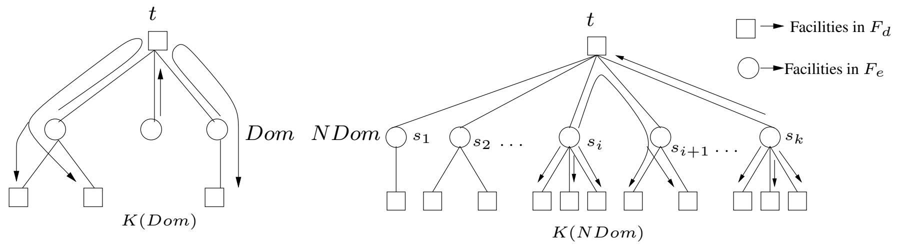


Figure 6.2: Reassigning the demand for the Single_Pivot, and Double_Pivot operations.

We consider the operation Double_Pivot($t, s_k, \Delta^1, \Delta^2$) in which we open $\{t\} \cup K(Dom) \cup K(s_k)$ and close $Dom \cup \{s_k\}$. We route the excess of Dom via t to $\{t\} \cup K(Dom)$. We route the excess of s_k via s_k to $\{t\} \cup K(s_k)$. In particular the vectors are set as follows. $\Delta^1_{i \in Dom} = u_i^* - u_i$ (this is negative as i shrinks or closes), $\Delta^1_{i \in K(Dom)} = u_i^* - u_i$, $\Delta^1_t = \sum_{i \in Dom} y_{i,t}$ (all these are positive) and zero for other facilities. The pivot for this vector is t. We set $\Delta^2_{s_k} = u^*_{s_k} - u_{s_k}$ (negative as s_k shrinks/closes), $\Delta^2_t = y_{s_k,t}$, $\Delta^2_{i \in K(s_k)} = u_i^* - u_i$ (positive), and zero for other facilities. We set s_k to be the pivot for this vector. It is easy to see that this is a feasible Double_Pivot operation.

Since we are opening t in the above operation, it has sufficient capacity to take a demand of $y_{i,t}$ for $i \in Ndom \setminus \{s_k\}$ as well. So, one could be tempted to reroute the excess of these facilities in $NDom$ by opening $K(NDom \setminus \{s_k\})$. But, if there is a facility $s_i \in K(NDom \setminus \{s_k\})$ such that $y_{s_i,K(s_i)}$ is arbitrarily larger than $y_{s_i,\{t,s_k\}}$ then, pivoting the flow $y_{s_i,K(s_i)}$ via either t or s_k could violate property P3 mentioned in previous section. So, just one Double_Pivot is not sufficient to bound the facility costs of $NDom \setminus s_k$. We bound the facility cost of the facilities in $Ndom \setminus \{s_k\}$ by defining a set of Single_Pivot operations.

For each facility $s_i \in NDom \setminus s_k$, we consider a Single_Pivot(s_i, Δ) operation. We set the capacity of s_i to its capacity in S^* (i.e, s_i is closed) and set the capacities of the facilities in $K(s_i) \cup K(s_{i+1})$ to at most capacities in S^* (i.e, open them). The vector Δ is defined as follows. For each facility $l \in K(s_i)$, $\Delta_l = y_{s_i,l}$. For each facility $l \in K(s_{i+1})$, $\Delta_l = y_{s_{i+1},l}$. For s_i, $\Delta_{s_i} = u^*_{s_i} - u_{s_i}$. The facility s_i is the pivot for this operation. Clearly, rerouting of the demand from s_i to $K(s_i)$ is feasible. By the sorted order of the facilities in $NDom$ and the fact that s_{i+1} is non-dominating means that $y_{s_i,t} \leq y_{s_{i+1},t} \leq y_{s_{i+1},K(s_{i+1})}$. Therefore, the routing of the the demand $y_{s_i,t}$ to $K(s_{i+1})$ is feasible and does not violate property P3 of the previous section.

This completes the set of operations performed on each subtree as defined in section 6.4.2. For a subtree $\mathcal{T}_t$, each facility in $K(t)$ is closed exactly once and each facility in $\{t\} \cup K(Dom)$ is opened exactly once. Each facility in $K(NDom)$ is opened exactly twice.

Lemma 6.4.4 *Each facility $i \in F_d$ is opened (i.e, its capacity is increased to u_i^*) at most three times.*

Proof. If a facility $r \in F_d$ is the root of a tree, then it is considered in operations of the subtree $\mathcal{T}_r$ only. Consider a facility $t \in F_d$ in a tree $\mathcal{T}$ rooted at $r \neq t$. Let $t' \in F_d$ be the facility such that $t \in K(K(t'))$. If t is a leaf, then it is considered in the operations of only one subtree $\mathcal{T}_{t'}$. Otherwise, it is considered in operations corresponding to two subtrees $\mathcal{T}_t$, and $\mathcal{T}_{t'}$. As a part of operations defined for $\mathcal{T}_{t'}$, it can be opened at most twice. As part of operations defined for $\mathcal{T}_t$, it is opened once. ∎

Let the flow across an edge $e \in \mathcal{F}$ in the optimal transshipment be given by y_e.

Lemma 6.4.5 *For each edge $e \in \mathcal{F}$, the amount of flow sent across e due to all the operations is at most three times the flow across it in the optimal transshipment (i.e, $3 \cdot y_e$).*

Proof. Let E_d denote the set of edges which connect dominant facilities to their children. Consider an edge $e \in E_d$. The Double_Pivot operation which sends a flow of y_e across e is the only operation which sends flow through e. Let E_n denote the edges which connect non-dominant facilities to their children. Consider an edge $e \in E_n$ connected to a non-dominant facility s_i. In the Single_Pivot operation pivoted at s_i, the flow sent along e is equal to y_e. In the Single_Pivot operation pivoted at s_{i-1}, the flow along e is at most y_e, thus bounding the flow along e by $2 \cdot y_e$. This also holds for edges connecting non-dominant facilities to their parent. Finally, consider the set of edges connecting dominant facilities to their parent. Let e be an edge which connects the root of a subtree $\mathcal{T}_t$ to its dominant facilities. The Double_Pivot operations sends the flow of $K(Dom)$ back and forth along e before sending them to $K(Dom)$ (Refer to Figure 6.2). As the flow to $K(Dom)$ is at most equal to the flow from Dom to t (i.e, y_e), this flow is at most $2 \cdot y_e$. The Double_Pivot also sends a flow of y_e to t. Thus, the flow along e due to the operations is at most three times its flow in the solution. As we have covered all edges, the lemma is true. ∎

Lemma 6.4.6 *The facility cost of the solution S is at most $C_f(S) \leq 3.C_f(S^*)+3(C_s(S)+C_s(S^*))$.*

Proof. The change in cost due to each of the operations is non-negative as S is a local optimum with respect to these operations. Let CHG denote the set of facilities whose capacities change because of a Single_Pivot(s, Δ) operation. It is easy to see that,

$$\sum_{i \in \text{CHG}} G_i(u_i^* - u_i) + \sum_{i \in \text{CHG}} c_{si}.|\Delta_i| \geq 0$$

Let CHG$_1$ and CHG$_2$ denote the set of facilities whose capacities change because of the two pivot operations in Double_Pivot$(s_1, s_2, \Delta^1, \Delta^2)$. It is easy to see that,

$$\sum_{i \in \text{CHG}_1 \cup \text{CHG}_2} G_i(u_i^* - u_i) + \sum_{i \in \text{CHG}_1} c_{s_1 i}.|\Delta_i^1| + \sum_{i \in \text{CHG}_2} c_{s_2 i}.|\Delta_i^2| \geq 0$$

From lemmas 6.4.4 and 6.4.5 it is clear that, adding equations of the form 6.4.3 and 6.4.3 for all the Single_Pivot and Double_Pivot operations, we get

$$3\sum_{i \in F_d}(G_i(u_i^*) - G_i(u_i)) + 3\sum_{s,t} c_{st} y_{s,t} \geq \sum_{i \in F_e} G_i(u_i^*) - G_i(u_i)$$

which gives us,

$$3C_f(S^*) + 3(C_s(S) + 3C_s(S^*)) \geq C_f(S).$$

From lemma 6.4.1, we have

$$6C_f(S^*) + 6C_s(S^*) = 6C(S^*) \geq C_f(S).$$

∎

Lemmas 6.4.1 and 6.4.6 imply the following theorem.

Theorem 6.4.7 *For any instance of UniFL, a locally optimum solution S with respect to add, Single_Pivot, and Double_Pivot operations has a total cost $C(S) \leq 7.C_f(S^*) + 7.C_s(S^*)$ where S^* is any optimal solution to the UniFL instance.*

Recently, we have proved that the techniques of Zhang et al. [59] can be combined with the Double_Pivot operation to show a locality gap of 6 for the UniFL problem [18]. Currently, this is the best known approximation factor for the UniFL problem.

Chapter 7

The budget constrained k-median problem

In this chapter, we consider the Budget Constrained k-median problem defined in Section 2.1.6. We propose a pre-processing algorithm followed by a local search heuristic. Our problem formulation helps us to study the effectiveness of local search for bicriteria optimization problems.

7.1 Motivation

The k-median, and k-center measures in facility location optimize contrasting objective functions. In the k-median problem, the goal is to minimize the average service cost of the set of clients. On the other hand, the k-center objective function is a fairness measure, i.e, it minimizes the maximum distance of a client to its nearest facility. So, while the challenge in the k-median problem is to minimize the L_1 norm, the challenge in the k-center problem is to minimize the L_∞ norm. This contrast in the objective functions can be exploited to show that a simultaneous approximation of both the measures within constant factors is not possible. Consider the example shown in Figure 7.1. A and B are points representing clusters of demand n. Z is a point with a unit demand. The distance between A and B is 1, while the distance of Z from both A and B is $\sqrt{n}$. Let k be 2. The optimal solution for the k-median measure picks A and B as the facilities and has a cost of $\sqrt{n}$. Its k-center cost is $\sqrt{n}$. The optimal solution for the k-center measure picks a facility at Z, and one at either A or B and has a cost of 1. Its k-median cost is n. So, any choice of 2 facilities exceeds the optimal value of one of the measures by a factor $\sqrt{n}$.

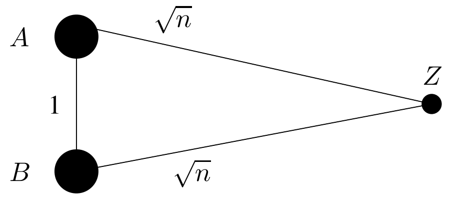


Figure 7.1: Example where approximating 2-median and 2-center within constant factors is not possible.

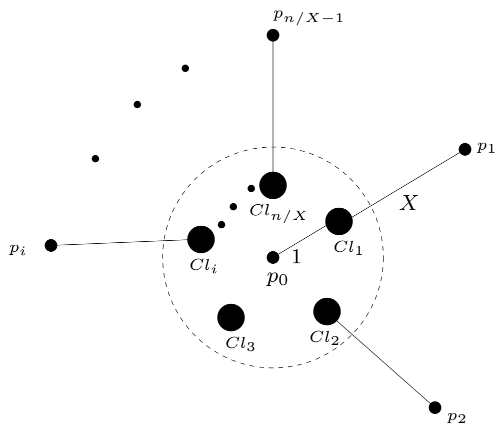


Figure 7.2: Sometimes medians can be fair to all clients.

However, there are situations when obtaining good solutions with respect to both k-median and k-center measures is possible and desirable. Consider the example shown in Figure 7.2. The figure shows n/X clusters $\{Cl_1, \ldots, Cl_{n/X}\}$ of size X which are at a distance of 1 from a central point p_0. There are $n/X - 1$ points $\{p_1, \ldots, p_{n/X-1}\}$ at a distance of X from one of the clusters as shown in the figure. Let k be n/X. Suppose we pick the center of all the clusters, namely, CH1 $= \{Cl_1, \ldots, Cl_{n/X}\}$. The k-median cost of CH1 is n and k-center cost is X. Instead, we could pick CH2 $= \{p_0, p_1, \ldots, p_{n/X-1}\}$ as the k facilities. The k-median cost of CH2 is also n. On the brighter side, the k-center cost of CH2 is 1. This example abstracts the problem of facility location in a city with many satellite towns around it. Although the natural answer in this case is quite intuitive, many computational problems with this requirement need specialized heuristics. With these motivations, we formulate the budget constrained k-median problem. In Section 2.1.6, we formulated the problem as follows.

- **Input:** The set of clients C. The distances between clients in C satisfy metric properties. Input also consists of an integer k and a *budget* B. For a client $j \in C$ and a set $S \subseteq C$, $ds(j, S)$ denotes $\min\limits_{i \in S} c_{ij}$.
- **Validity:** A *valid solution* $S \subseteq C$ is such that, $(|S| \leq k) \wedge (\forall\, i \in C : ds(i, S) \leq B)$.
- **Assumption:** The input has at least one valid solution.
- **Output:** A valid solution S such that $\sum_{i \in C} ds(i, S)$ is minimized.

We have surveyed the existing results on the k-median problem in Chapter 3. Gonzales [19] gave a farthest point greedy heuristic for the k-center problem, which runs in $O(nk)$ time and gives a 2-approximation. Hochbaum and Shmoys [24] also gave a 2-approximation for this problem using a generic technique applicable to a class of “bottleneck” problems. They also show that no polynomial time algorithm can approximate the k-center problem to a factor better than 2 unless $P = NP$. The primal-dual based Lagrangian relaxation technique of Jain and Vazirani [27] can be modified to get a $(6, 9)$ approximation algorithm for the budget constrained k-median problem. Essentially, the input is pruned to contain only those edges whose lengths are within the budget. However, we study this problem to show that the local search technique can be used in bi-criteria optimization problems with an appropriate pre-processing stage. The budget represents a set of constraints on the service cost of each client. Local search techniques for constraint programming (CP) problems with optimization goals have also been studied. Travelling salesman problem with time windows (TSPTW) is a typical example of such problems. Pesant and Gendreau [48] integrate branch and bound technique with local search to report empirical results on the TSPTW problem. Our technique gives rise to the question of applying local search in CP problems using appropriate pre-processing techniques.

Our algorithm has two phases: a preprocessing phase and a local search phase. In the preprocessing phase, we partition the set of clients C into mutually disjoint sets $C_1, C_2, \dots C_l$ such that the diameter of each C_i is within a constant times the budget B. Furthermore, any valid solution is guaranteed to open a certain minimum number of facilities inside each of $C_1, \dots, C_l$. In the local search space, we run the single swap heuristic for the k-median problem with the restriction that it explores only those solutions which open at least one facility in each of the C_i's. We call it as *restricted local search* heuristic. We call a solution $S \subseteq C$ as *fine* with respect to the partition, if it opens at least one facility in each of the C_is. Our goal is to use the analysis technique presented in Section 3.3 to the modified procedure to obtain a constant factor approximation for the k-median measure. In this chapter, we show that such an approach is feasible.

7.2 A simple scheme that fails

As there is atleast one valid solution, we consider the following iterative algorithm to cluster the set of clients. In iteration i, pick an unclustered client $j \in C$. Create a new cluster C_i consisting of j and all unclustered clients within a distance of $2 \cdot B$ from j. The proof of the factor 2 approximation algorithm for the k-center given by Hochbaum and Shmoys [24] guarantees that we get atmost k clusters. Let these clusters be $C_1, \dots, C_l, l \leq k$.

Let us consider the restricted local search with $C_1, \dots, C_l$ as the partition of the client set. Let S denote a local minimum computed by the restricted local search. Let O denote the optimal solution for the budget constrained k-median problem. Note that, our construction ensures that every client has an open facility within a distance of $4 \cdot B$. We would like to apply the analysis of k-median problem presented in Section 3.3 to bound the service cost of S. The highlights of the k-median analysis were:

P1. Swap every 1-bad facility with the optimum facility that it captures.

P2. Swap the rest of the optimum facilities with good facilities such that each good facility is considered in atmost two swaps.

However, in case of restricted local search, we can consider only those swaps which result in a fine solution with respect to $C_1, \ldots, C_l$. This restriction prevents us from considering the following swaps which were essential in the k-median analysis. As always, we denote the local optimum solution by S and the optimum by O.

B1. Consider a 1-bad facility $s \in S$ which captures $o \in O$. Suppose s is the only facility opened inside a cluster C_i and $o \notin C_i$ then, the swap $\langle s, o \rangle$ is not allowed as the resulting solution is not a fine solution.

B2. Consider a good facility $s \in S$ which is the only facility opened inside a cluster C_i. Clearly, s can be swapped with only those facilities in optimum which are inside C_i.

The limitations B1 and B2 prevent us from applying the analysis of the k-median problem to bound the service cost of S in terms of the service cost of O. In fact, the idea cannot be made to work even by computing the clusters such that any valid solution is guaranteed to open at least 2 facilities in each C_i. We first observe that a set of swaps which satisfy the following relaxed constraints is sufficient to obtain a constant factor approximation:

P3. All the optimum facilities can be considered in exactly one swap while ensuring that no facilities in S is considered in more than a constant number of swaps.

P4. If a swap $\langle s, o \rangle$ is considered, then the facility s does not capture any facility $o' \neq o$.

The observations made in B1, B2, P3, and P4 motivate us to compute a *bounded diameter partition*. Informally, we compute a partition in which either a cluster is far from other clusters or any valid solution is guaranteed to open atleast three facilities inside it. In the rest of our discussion, we use the terms cluster and region interchangeably. The diameter of each region is bounded by constant times the budget B. We formalize these ideas in the next section.

7.3 Computing a Bounded Diameter Partition

In this section, we give a procedure which computes a bounded diameter partition. This construction is used to formulate a generalized notion of bounded diameter partition. We first try to cover all the clients by regions in which any valid solution is guaranteed to open at least three facilities. The clients not covered by such regions are covered by regions which are far from others. We choose our parameters such that a local minimum of restricted local search has at most $k/3$ limitations of type

B1 and B2. Our partitioning is iterative and begins with all the clients classified as "uncovered". We use R's to denote clusters or regions which are essentially subsets of the set of clients. We use the term *envelope of radius r around a region R* to denote the set of clients $\{j \in C | ((j \notin R) \wedge (ds(j, R) \leq r))\}$. For a client $j \in C$, $\mathcal{B}(j, r)$ denotes the set of all clients within a distance of r from j.

7.3.1 Triplet Phase

Initially, we form regions in which any valid solution is guaranteed to open at least 3 facilities. We call such a region as *triplet region*. A triplet region T is defined by three clients p_1, p_2, and p_3 such that $\mathcal{B}(p_1, B)$, $\mathcal{B}(p_2, B)$, $\mathcal{B}(p_3, B)$ are disjoint and one of the points, say p_1, is at a distance of at most $6 \cdot B$ from both p_2, p_3. Figure 7.3 shows a triplet region. The set of clients assigned to a triplet region is a subset of the set of clients in the union of balls of radius $3 \cdot B$ around $p_1, p_2,$ and p_3. The set of all clients in the balls of radius B around p_1, p_2, and p_3 is said to be the *core* of such a triplet region and denoted by $Core(T)$. So, the maximum distance of a client inside a triplet region from its core is at most $16 \cdot B$. A maximal set of triplet of regions can be computed iteratively. In each

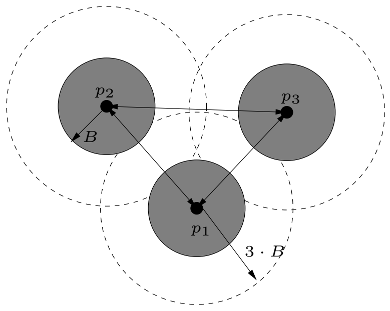
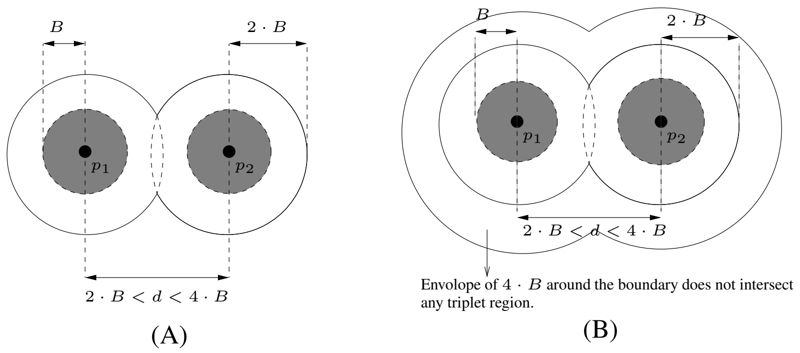


Figure 7.3: Illustration of a triplet region

iteration, three uncovered clients which can form a triplet region are identified. Suppose p_{i1}, p_{i2}, p_{i3} are the points satisfying triplet condition in iteration i. We mark all the uncovered clients inside $\cup_{q \in \{1,2,3\}} \mathcal{B}(p_{iq}, 3 \cdot B)$ as covered and assign it to the ith triplet region, T_i. Note that, some of the clients in $\cup_{q \in \{1,2,3\}} \mathcal{B}(p_{iq}, 3 \cdot B)$ may have been assigned to triplet regions formed in earlier iterations. Also, note that any of p_{i1}, p_{i2}, p_{i3} could be close to a triplet region formed in previous iterations. So, $Core(T_i)$ need not be a subset of T_i. We terminate the process when it is not possible to find three uncovered clients which satisfy triplet condition. It is easy to see that if T_1 and T_2 are two triplet regions identified by this iterative process, then $Core(T_1) \cap Core(T_2) = \emptyset$. We call this phase as the *triplet phase*. Clearly, it can be implemented in $O(kn^3)$ time.

7.3.2 Doublet Phase

Note that the triplet phase may not cover all the clients. We try to cover the remaining clients by regions in which any valid solution is guaranteed to open at least 2 facilities called *doublet region*. A doublet region D is defined by two clients p_1, and p_2 such that $2B < ds(p_1,\, p_2) < 4 \cdot B$ and there does not exist an uncovered client p_3 such that $p_1,\ p_2$, and p_3 form a triplet region. The set of clients assigned to a doublet region is a subset of clients in the union of balls of radius $2 \cdot B$ around p_1, and p_2. Figure 7.4(A) shows a doublet region.



B
$2 \cdot B$
p_1
p_2
$2 \cdot B < d < 4 \cdot B$
(A)
B
$2 \cdot B$
p_1
p_2
$2 \cdot B < d < 4 \cdot B$
Envelope of $4 \cdot B$ around the boundary does not intersect any triplet region.
(B)

Figure 7.4: Illustration of a doublet region

Lemma 7.3.1 *After the triplet phase, the envelope of radius $4 \cdot B$ around a doublet region defined by two uncovered clients does not contain any uncovered clients.*

Proof. Suppose an uncovered client p_3 exists in the envelope of radius $4 \cdot B$ around the doublet region defined by p_1, and p_2. The clients p_1, p_2, and p_3 define a triplet region and the set of triplet regions formed by triplet phase is not maximal. This is contrary to the fact that the triplet phase terminated. Hence, the envelope of $4 \cdot B$ around any doublet region is free of uncovered clients. ∎

Let D_1 be a doublet region defined by two uncovered clients (p_1, p_2) after the triplet phase. Let D_2 be a doublet region defined by two uncovered clients (p_3, p_4) such that $\{p_1, p_2\} \cap \{p_3, p_4\} = \emptyset$. Let U_1 denote the set of uncovered clients in the union of balls of radius $2 \cdot B$ around p_1, p_2. Let U_2 be defined similarly for p_3, p_4. The proof for Lemma 7.3.1 implies the following lemma.

Lemma 7.3.2 *The distance of the two sets U_1, U_2 defined by* $\max_{i \in U_1, j \in U_2}(c_{ij})$ *is at least* $4 \cdot B$.

We now form a set of doublet regions in an iterative manner. At each iteration, we find two points p_1, p_2 which define a doublet region. Let D be the set of uncovered clients in the union of balls of radius $2 \cdot B$ around p_1, p_2. If the envelope of radius $4 \cdot B$ around D intersects a triplet region T discovered in the triplet phase, then all the clients in D are marked covered and assigned to T. If the envelope intersects many triplet regions, then one of them is chosen arbitrarily. If the envelope does not intersect any of the triplet regions, then the doublet region is called an *isolated doublet region* (Figure 7.4(B)). All the clients in D are marked covered and assigned to the isolated doublet region. The core of an isolated doublet region denoted by $Core(D)$, contains all the clients assigned to it. Lemma 7.3.2 implies that no two doublet regions are attached to a triplet region through a chain. In other words, if D_1 and D_2 are two doublet regions which got merged with a triplet region T formed in triplet phase, then both D_1 and D_2 intersect the envelope of radius $4 \cdot B$ around T. This phase is called the *doublet phase*.

7.3.3 Singlet Phase

So far, we have covered a subset of the set of clients by triplet regions, and isolated doublet regions. We cover the remaining uncovered clients in the following manner. After the doublet phase, the distance between two uncovered clients is either less than $2 \cdot B$ or greater than $4 \cdot B$. Each set of uncovered clients which are at a distance of less than $2 \cdot B$ from each other form a *singlet region* as shown in Figure 7.5.

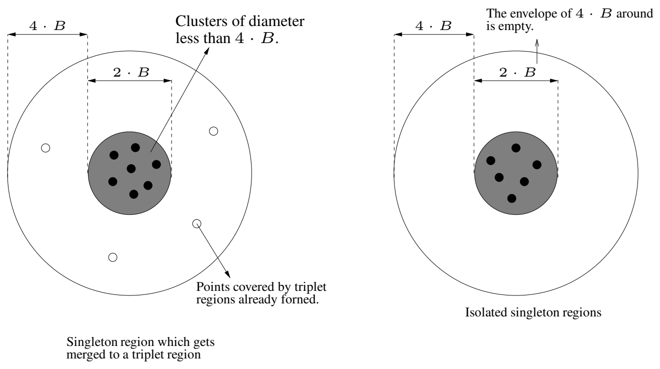

$4 \cdot B$
$2 \cdot B$
Clusters of diameter less than $4 \cdot B$.
Points covered by triplet regions already forned.
Singleton region which gets merged to a triplet region
$4 \cdot B$
$2 \cdot B$
The envelope of $4 \cdot B$ around is empty.
Isolated singleton regions

Figure 7.5: Illustration of a singlet region

Lemma 7.3.3 *After the doublet phase, the envelope of radius $4 \cdot B$ around a singlet region does not intersect an isolated doublet region or a doublet region which got merged with a triplet region.*

Proof. Similar to the proof of 7.3.1. ∎

However, the envelope of radius $4 \cdot B$ around a singlet region may intersect a triplet region T. If such is the case, then we mark all the clients in the singlet region as covered and assign to the triplet region T. Otherwise, such a singlet region is called an *isolated singlet region.* All the clients inside an isolated singlet region are covered and assigned to it. The core of an isolated singlet region is the set of all clients assigned to it. Thus, we have covered all the clients by assigning to triplet regions, isolated doublet regions, and isolated singlet regions.

Lemma 7.3.4 *The partition obtained by the above process is such that, the maximum distance of a client from the core of its region is at most $28 \cdot B$.*

Proof. The upper bound on the diameter of a region is the diameter of a region formed by merging a doublet region with a triplet region. The diameter of a triplet region in the triplet phase with respect to its core is $16 \cdot B$. The diameter of a doublet region is $8 \cdot B$ and they can be separated by at most $4 \cdot B$. Thus, the diameter of a region with respect to its core is bounded by $28 \cdot B$. ∎

7.3.4 Generalization

We have shown how to construct a partition in which the diameter of each region with respect to its core is at most $28 \cdot B$. Each region is either a triplet region, or an isolated region. The cores of any two regions are disjoint. This idea can be generalized to define (a, b) partition where a is a constant greater than 2 and b is the maximum diameter of a region with respect to its core.

Consider an input (C, k, B) to the budget constrained k-median problem, where C is the set of clients, k is an integer, and B is the budget on the quality of the k-center solution. For a client j, we denote the set of all clients within a distance of r by $\mathcal{B}(j, r)$. We now define a (a, b)*-partition* of the set of clients. Intuitively, the idea is to partition the set of clients into regions of radius at most $b \cdot B$. Furthermore, for each region, any valid solution is guaranteed to open a facilities inside it, or it is far from other regions. Formally, a partition of the set of clients C into *region*s $R_1, \ldots, R_l$ is called (a, b)-partition if the following properties hold:

1. $R_p \neq \emptyset$ and $R_p \cap R_q = \emptyset$ for all $p \neq q$,

2. For a region R_p ($p = 1, \dots, l$),

   (a) either, we can fix a set $R_p^a = \{j_1, \dots j_a\} \subseteq R_p$ such that, if $((j_x, j_y \in R_p^a) \wedge (j_x \neq j_y))$, then $\mathcal{B}(j_x, B) \cap \mathcal{B}(j_y, B) = \emptyset$. Such a region is called *a-populated region*. The set $\cup_{j_x \in R_p^a} \mathcal{B}(j_x, B)$ is said to be the *core* of the a-populated region. Some of the clients in the core of an a-populated region may belong to different regions.

   (b) or, for any $j \in C$, $j \notin R_p$ implies $ds(j, R_p) = \min_{j' \in R_p} ds(j, j') > 4 \cdot B$. Let $\{j_1, \dots j_w : w < a\} \subseteq R_p$ be the maximum number of points such that $\mathcal{B}(j_x, B)$'s are disjoint. This means that, for any client $j \in R_p$, $ds(j, \{\mathcal{B}(j_1, B), \dots, \mathcal{B}(j_w, B)\}) \leq 2 \cdot B$. We call such a region as w-populated-*isolated* region. The core of an isolated region is the entire region.

3. Furthermore, if R_p, R_q are two a-populated regions defined by R_p^a, R_q^a, then $ds(R_p^a, R_q^a) > 2 \cdot B$. This implies that the core of two a-populated regions do not intersect.

4. The maximum distance of a client from the core of its region is at most $b \cdot B$.

Properties 2(a) and 4 ensure that any valid solution has to open at least a facilities inside the core of an a-populated region. The preprocessing algorithm of the previous section gives a a $(3, 28)$ partition. It can be generalized to obtain a $(K, 10K - 2)$ partition for any constant K by suitably defining K-plet regions. The most straightforward implementation of K-plet regions will take $O(kn^K)$ time.

7.4 Restricted Local Search

The single-swap heuristic for the k-median problem presented in Section 3.3, starts with an arbitrary set of size k and at each step considers doing a swap which improves the cost. The algorithm terminates when a local minima is reached. Let $R_1, \dots, R_l$ denote the bounded diameter partition computed by the preprocessing phase. The restricted local search mimics the k-median algorithm while exploring only fine solutions, i.e, atleast one facility is opened inside each region. Refer to Figure 7.6.

As always, $cost(S)$ denotes the service cost of a solution S. An algorithm is said to be a (α, β)-approximation if, for every instance of the problem, it opens a set of k facilities, denoted by S, such that, the $cost(S)$ is at most α times the cost of the optimal valid solution and the induced k-center cost of S is at most βB. The main theorem we want to prove is:

```
    Algorithm     Restricted Local Search.

1.  $S \leftarrow$ an arbitrary fine solution.
2.  While $\exists\, s \in S$ and $s' \notin S$ such that $S - \{s\} + \{s'\}$ is a fine solution
          and $cost(S - \{s\} + \{s'\}) < cost(S)$,
    do $S \leftarrow S - \{s\} + \{s'\}$.
3.  return $S$.
```

Figure 7.6: Restricted Local Search

Theorem 7.4.1 *If the restricted local search algorithm in Figure 7.6 uses (a, b)-partition, it is a $(((5a - 2)/(a - 2)) + \epsilon, b)$-approximation algorithm for budget constrained k-median problem.*

7.5 Analysis of Locality Gap

In this section, we prove the theorem stated in 7.4.1. Here, we assume that a suitable (a, b)-partition is obtained, for some $a > 2$. We analyze the locality gap of the restricted local search. As always, the local optimum solution output by the restricted local search is denoted by S and O denotes an optimum solution. We reuse all the notations introduced in the analysis of the k-median problem and consider a 1-1 and onto function $\pi : N_O(o) \rightarrow N_O(o)$ as defined in Section 3.3 for each facility $o \in O$.

The following simple observation can be made about a facility $s \in S$ if it is the only facility opened inside an isolated region.

Lemma 7.5.1 *If a facility $s \in S$ is the only facility opened inside an isolated region R_i, then it has to capture at least one facility $o \in O$ which is inside R_i.*

Proof. Suppose $s \in S$ is the only facility inside a x-populated isolated region R_i. Any valid solution has to open a facility within a distance of B from s, say $o \in O$. By definition of isolated region, $o \in R_i$. So, all the clients inside $N_O(o)$ are within a distance of $2 \cdot B$ from s. Since the region R_i is isolated, if $c \notin R_i$, then its distance from s is greater than $4 \cdot B$. So, its distance from clients in $N_O(o)$ is greater than $2 \cdot B$. So, s is the nearest facility to all clients in $N_O(o)$ and captures o. ∎

The main idea is to define a set of k swaps such that, each swap results in a fine solution. We then use our standard technique of proving the locality gap by writing inequalities corresponding to each swap. As in the analysis of the k-median heuristic, we would like to swap a 1-bad facility with the optimum facility that it captures. Also, we do not include a (2+)-bad facility in any swap. We classify the facilities in S depending on the restrictions they impose on considering single swaps.

Consider a facility $s \in S$ which captures exactly one facility $o \in O$. If s is the only facility inside the core of an a-populated region R_i and o is not in the core of R_i, then the swap $\langle s, o \rangle$ is not feasible as the resulting solution is not a fine solution. Such a facility s is called *constrained* 1-bad facility. Let B_1^c denote the set of constrained 1-bad facilities. Let $b_1^c = |B_1^C|$. The rest of the 1-bad facilities are called *unconstrained* 1-bad and denoted by B_1^u. Let $b_1^u = |B_1^u|$. We use B_{2+} to denote the set of $(2+) - bad$ facilities. Let $b_{2+} = |B_{2+}|$.

Consider a good facility $s \in S$. Suppose it is the only facility opened by S inside the core of a region R_i. Clearly, s can be considered can be swapped with only those facilities in the optimum which are inside the core of the region R_i. We call such a good facility as *constrained good* facility and denote the set of constrained good facilities by G_c. Let $g^c = |G^c|$. Rest of the good facilities are called *unconstrained good* facilities and can be considered in any swap and denoted by G^u. Let $g^u = |G^u|$.

By lemma 7.5.1, constrained 1-bad facilities and constrained good facilities can be present in a-populated regions only. We consider the following swaps.

- Suppose s is an unconstrained 1-bad facility capturing o, then we consider the swap $\langle s, o \rangle$.
- The remaining $k - b_1^u$ facilities of O are considered in swaps with only unconstrained good facilities. Each unconstrained good facilities is considered in $(k - b_1^u)/g^u$ swaps.

The swaps considered above satisfy the following properties.

1. Each $o \in O$ is considered in exactly one swap.
2. A facility $s \in S$ which captures more than one facility in O is not considered in any swap.
3. Each good facility $s \in S$ is considered in at most $(k - b_1^u)/g^u$ swaps.
4. If swap $\langle s, o \rangle$ is considered then facility s does not capture any facility $o' \neq o$.

We claim that $g^u \geq \frac{(k-b_1^u)(a-2)}{2a}$. Suppose this is true, then $(k - b_1^u)/g_u \leq 2a/(a-2)$. The proof of our claim follows.

After we consider the swaps of all unconstrained 1-bad facilities, we are left with $k_1 = k - b_1^u$ optimum facilities to be considered for swaps. The number of constrained 1-bad facilities, and constrained good facilities is bounded by the maximum number of a-populated regions required to cover k_1 facilities.

$$g^c + b_1^c \le k_1/a \tag{7.1}$$

As we are left with exactly k_1 facilities in our solution which have not been considered so far:

$$g^c + g^u + b_1^c + b_{2+} = k_1 \tag{7.2}$$

Clearly, the number of facilities captured by *bad* facilities cannot exceed k_1.

$$b_1^c + 2b_{2+} \le k_1 \tag{7.3}$$

Simple manipulations show that:

$$\begin{aligned}
2g^c + 2g^u + 2b_1^c + 2b_{2+} &= 2k_1\{\text{multiplying (7.2) by 2 }\} \\
2g^c + 2g^u + b_1^c &\ge k_1\{\text{by substituting (7.3) }\} \\
g^c + 2g^u &\ge \frac{k_1(a-1)}{a}\{\text{ Substituting (7.1) }\} \\
2g^u &\ge \frac{k_1(a-2)}{a}\{\text{Substituting (7.1) }\} \\
g_u &\ge \frac{k_1(a-2)}{2a} \qquad (7.4)
\end{aligned}$$

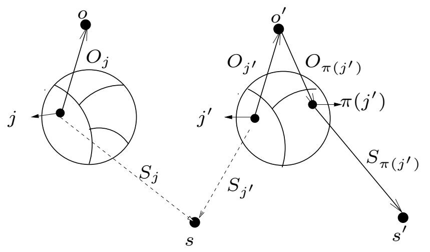


Figure 7.7: Reassigning the clients in $N_S(s) \cup N_O(o)$.

We now analyze these swaps one by one. Consider a swap $\langle s, o\rangle$. We place an upper bound on the increase in the cost due to this swap by reassigning the clients in $N_S(s) \cup N_O(o)$ to the facilities in $S - s + o$ as follows (Figure 7.7). The clients $j \in N_O(o)$ are now assigned to o. Consider a client $j' \in N_s^{o'}$, for $o' \neq o$. As s does not capture o', by the property of our mapping function π, we have that $\pi(j') \notin N_S(s)$. Let $\pi(j') \in N_S(s')$. Note that the distance that the client j' travels to the nearest facility in $S-s+o$ is at most $c_{j's'}$. From triangle inequality, $c_{j's'} \leq c_{j'o'}+c_{\pi(j')o'}+c_{\pi(j')s'} = O_{j'} + O_{\pi(j')} + S_{\pi(j')}$. The clients which do not belong to $N_S(s) \cup N_O(o)$ continue to be served by the same facility.

Therefore,

$$\sum_{j \in N_O(o)} (O_j - S_j) + \sum_{\substack{j \in N_S(s), \\ j \notin N_O(o)}} (O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) \geq 0. \tag{7.5}$$

As each facility $o \in O$ is considered in exactly one swap, the first term of inequality (7.5) added over all k swaps gives exactly $cost(O) - cost(S)$. For the second term, we will use the fact that each $s \in S$ is considered in at most $2a/(a-2)$ swaps. Since S_j is the shortest distance from client j to a facility in S, using triangle inequality we get: $O_j + O_{\pi(j)} + S_{\pi(j)} \geq S_j$. Thus the second term of inequality (7.5) added over all k swaps is no greater than $2a/(a-2)\sum_{j\in C}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j)$. But since π is a 1-1 and onto mapping, $\sum_{j\in C} O_j = \sum_{j\in C} O_{\pi(j)} = cost(O)$ and $\sum_{j\in C}(S_{\pi(j)} - S_j) = 0$. Thus, $2a/(a-2)\sum_{j\in C}(O_j + O_{\pi(j)} + S_{\pi(j)} - S_j) = 4a/(a-2) \cdot cost(O)$. Combining the two terms we get, $cost(O) - cost(S) + 4a/(a-2) \cdot cost(O) \geq 0$. Thus, $(5a-2)/(a-2) \cdot Cost(O) \geq Cost(S)$. This proves the theorem 7.4.1. ∎

Theorem 7.5.2 *There exists a $(13+\epsilon, 28)$ approximation algorithm for the budgeted k-median problem.*

Proof. The construction given in Section 7.3 gives a $(3, 28)$ partition. From Theorem 7.4.1, it follows that there exists a $(13+\epsilon, 28)$ approximation algorithm for the budgeted k-median problem. ∎

More generally, using a preprocessing stage which computes K-plet regions, we can get a $(5K - 2/K - 2, 10K - 2)$ approximation algorithm for any constant K.

Chapter 8

Conclusions and open problems

In this thesis, we showed the power of local search for approximating facility location problems. Local search is widely used by programmers for their simplicity in understanding and implementing. The results shown here go towards explaining the success of local search heuristics. The study of local search for different combinatorial optimization is desirable not just to obtain better algorithms, but also to better understand their relation with complexity of computational problems.

The algorithms presented in this thesis have the same iterative structure. The local operations vary depending on the problem. The set of local operations needed to prove the locality gaps for different problems also bear a strong resemblance to the way the heuristics are strengthened when different pathological cases emerge. The approximation ratio achieved for the k-median problem remains the best known. So far, local search is the only known technique to yield non-trivial approximation ratios for capacitated versions with hard capacities. We also present a local search heuristic for the budget constrained k-median problem which opens the possibility of local search heuristics for bi-criteria and constraint satisfaction optimization problems.

Along the line of the results presented in this thesis, several possibilities emerge. Most important among them is, does there exist a characterization of the games for which the price of anarchy can be analyzed via locality gap? Local search analyzes of combinatorial optimization problems with different covering/packing constraints may help obtain better characterization of complexity classes as presented in Section 2.3. Can the power of local search be improved by intelligent preprocessing as shown in Chapter 7?

In the context of facility location problems, several challenges remain open. They are:

1. Most important open problem in facility location is the *generalized median problem*. In the

generalized median problem, the facility costs of the facilities are non-uniform. There is a fixed budget available for opening the facilities. The goal is to open a set of facilities whose total cost is within the budget and the service cost is minimized. The k-median problem is a special case of the generalized median problem in which the facility costs are uniform. No non-trivial approximation ratio is known for this problem. As in the case of capacitated facility location with hard capacities, it would be interesting to see if a local search technique gives the first approximation algorithm for this problem.

2. Another important problem in facility location is to get better estimates of lower bounds for capacitated facility location instances via linear programs. This may eventually help to design approximation algorithm for capacitated facility location with hard capacities based on linear programming. Some preliminary results in this direction dealing with special cases of capacitated facility location with hard capacities is presented by Levi, Shmoys, and Swamy [36].

3. Approximation algorithms for the capacitated versions of the k-median problem, fault tolerant facility location [22, 54], and facility location with outliers [11] would also be interesting.

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Bio-data

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