In this lecture we would study about Chordal Graphs.

2.1 Induced Subgraph

Definition 2.1 Let $G = (V, E)$ be a Graph.

Let $V' \subseteq V$ be a subset of vertices of $G$.

The subgraph of $G$ induced by $V'$ is the subgraph $G' = (V', E')$ of $G$ that has $E' = E \cap (V' \times V')$.

That is, it contains all the edges of $G$ that connect elements of the given subset of the vertex set of $G$ and only those edges.

2.2 Chordal Graphs

Definition 2.2 A Chordal Graph is a graph that does not contain an induced cycle of length greater than 4.

In other words, it is a graph in which every cycle of length four and greater has a cycle chord.

Figure 2.1: A chordal graph
Theorem 2.3 A graph $G$ is chordal iff it has a perfect elimination ordering.

Proof: The easy part is to show that if $G$ has a perfect elimination ordering, then it is chordal. Suppose, for contradiction, that this is false. Let $G$ be a graph with a perfect elimination ordering and suppose there is a chordless cycle $v_1, v_2, \ldots, v_l$ of length $l \geq 4$ in $G$. Let $v_i$ be the vertex in the cycle that occurs first in the perfect elimination ordering. Then $v_{i-1}$ and $v_{i+1}$ are neighbors of $v_i$ in $G$ that occur later in the ordering. Since the ordering is perfect, there must be an edge between $v_{i-1}$ and $v_{i+1}$, but this contradicts the assumption that the cycle is chordless.

Now, show the converse, that if $G$ is chordal then it has a perfect elimination ordering. For that we would need the concept of separators.

Definition 2.4 A separator is a partition $V = S \cup A \cup B$ of the vertices such that there are no edges between $A$ and $B$.

Definition 2.5 Given two non-adjacent vertices $a$ and $b$, an $(a, b)$-separator is a separator $V = S \cup A \cup B$ such that $a \in A$ and $b \in B$.

![Figure 2.2: A $(a, b)$-separator](image)

Definition 2.6 Given two non-adjacent vertices $a$ and $b$, a minimal $(a, b)$-separator is an $(a, b)$-separator $V = S \cup A \cup B$ such that no subset of $S$ is an $(a, b)$-separator.

Definition 2.7 A simplicial vertex of a graph $G$ is a vertex $v$ such that the neighbors of $v$ form a clique in $G$.

Lemma 2.8 Given a chordal graph $G = (V, E)$ and two vertices $a, b \in V$ such that $(a, b) \notin E$, any minimal $a$-$b$ separator is a clique.

Proof: We would prove this by contradiction. Let $S$ be a minimal $(a, b)$-separator. For any vertex set $T$, let $G_T$ be the graph induced by $T$. Then $G_{V-S}$ has a number of connected components; one contains $a$ (let those vertices be $A$), one contains $b$ (let those vertices be $B$), and there may be other connected components. Consider any two vertices $x, y$ in the minimal a-b separator $S$ and suppose that $(x, y) \notin E$.

Note first that $x$ must have a neighbor $a_x$ in $A$, for otherwise $S - x$ would also be an $(a, b)$-separator, contradicting the minimality of $S$. Likewise, $y$ has a neighbor $a_y$ in $A$. 
Since $G_A$ is connected, there is a path from $a_x$ to $a_y$ using only vertices in $A$. Thus, there exists a path from $x$ to $y$ for which all intermediate vertices are in $A$. Among all such paths, let the shortest one be $x, a_1, a_2, \ldots, a_k, y$ and note that it has length at least 2 since $x$ and $y$ are not adjacent. Similarly we can find a shortest path from $x$ to $y$ for which all intermediate vertices are in $B$.

![Figure 2.3: Minimal (a,b)-separator is a clique](image)

Combining the two paths yields a cycle of length at least 4, which must have a chord since $G$ is chordal. However, there is no chord in the cycle from $x$ or $y$ to either $A$ or $B$ since we chose the shortest paths from $x$ to $y$ in each component. Neither is there an edge from $A$ to $B$ since $A$ and $B$ are two different components. The only other possibility is for there to be a chord between $x$ and $y$, but $x$ and $y$ are not adjacent. So we have a contradiction, which means that $(x, y) \in E$.

Clearly, if $G$ has a perfect elimination order, then the last vertex in it is simplicial in $G$. This gives rise to a simple algorithm to find a perfect elimination order if one exists:

**Algorithm:** Find perfect elimination order.
For $i = n, \ldots, 1$
Let $G_i$ be the graph induced by $V_{v_i+1}, \ldots, v_n$.
Test whether $G_i$ has a simplicial vertex $v_i$.
If no, then stop. $G_i$ (and therefore $G$) has no perfect elimination order.
Else, set $v_i = v_i$.
$v_1, \ldots, v_n$ is a perfect elimination order.

Note that if $G$ is chordal, then after deleting some vertices, the remaining graph is still chordal. So in order to show that every chordal graph has a perfect elimination order, it suffices to show that every chordal has a simplicial vertex; the above algorithm will then yield a perfect elimination order.

Now we show that every chordal graph has a simplicial vertex. In fact, we show a slightly stronger statement, which is needed for the induction hypothesis.

**Lemma 2.9** A connected chordal graph is either a clique, or it contains two non adjacent simplicial vertices.

**Proof:** If $G$ is chordal and it is a clique we are done. Assume that it is not a clique. Therefore we have two non-adjacent vertices $a, b$ in $G$. Consider the minimal $a - b$ separator, $S$.

Induction on $A \cup S$ (refer to the definition above).
If it is a clique then $a$ is a simplicial vertex or it has two non adjacent simplicial vertices $a_1$ and $a_2$, both of which cannot lie in $S$ as $S$ is a clique. Therefore either $a_1$ or $a_2$ lie in $A$. Similarly we can find a second non adjacent simplicial vertex when we consider $B$. ■
2.3 Independent Set

The maximal independent set problem is a NP-Hard Problem on general graphs but on graphs having a partial elimination ordering this problem can be solved efficiently.

Claim 2.10  There is an efficient algorithm to solve the independent set problem on graphs with a partial elimination ordering.

Proof: Algorithm: Scan the vertices in order, and for each $v_i$, add $v_i$ to $I$ if none of its predecessors has been added to $I$.
Scan Order: Let $v_1, v_2, \ldots, v_n$ be a perfect elimination order. Then the greedy algorithm applied with order $v_n, v_{n-1}, \ldots, v_1$ gives a maximum independent set.