5

Colouring

How many colours do we need to colour the countries of a map in such a way that adjacent countries are coloured differently? How many days have to be scheduled for committee meetings of a parliament if every committee intends to meet for one day and some members of parliament serve on several committees? How can we find a school timetable of minimum total length, based on the information of how often each teacher has to teach each class?

vertex colouring

A *vertex colouring* of a graph $G = (V, E)$ is a map $c: V \to S$ such that $c(v) \neq c(w)$ whenever v and w are adjacent. The elements of the set S are called the available *colours*. All that interests us about S is its size: typically, we shall be asking for the smallest integer k such that G has a *k-colouring*, a vertex colouring $c: V \to \{1, \ldots, k\}$. This k is the *(vertex-) chromatic number* of G; it is denoted by $\chi(G)$. A graph G with $\chi(G) = k$ is called *k-chromatic*; if $\chi(G) \leqslant k$, we call G *k-colourable*.

chromatic number $\chi(G)$

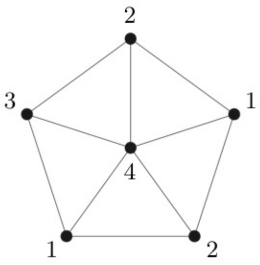

Fig. 5.0.1. A vertex colouring $V \to \{1, \ldots, 4\}$

colour classes

Note that a k-colouring is nothing but a vertex partition into k independent sets, now called *colour classes*; the non-trivial 2-colourable graphs, for example, are precisely the bipartite graphs. Historically, the colouring terminology comes from the map colouring problem stated above, which leads to the problem of determining the maximum chromatic number of planar graphs. The committee scheduling problem, too, can be phrased as a vertex colouring problem—how?

edge colouring

chromatic index $\chi'(G)$

An *edge colouring* of $G = (V, E)$ is a map $c\colon E \to S$ with $c(e) \neq c(f)$ for any adjacent edges e, f. The smallest integer k for which a *k-edge-colouring* exists, i.e. an edge colouring $c\colon E \to \{1, \dots, k\}$, is the *edge-chromatic number*, or *chromatic index*, of G; it is denoted by $\chi'(G)$. The third of our introductory questions can be modelled as an edge colouring problem in a bipartite multigraph (how?).

Clearly, every edge colouring of G is a vertex colouring of its line graph $L(G)$, and vice versa; in particular, $\chi'(G) = \chi(L(G))$. The problem of finding good edge colourings may thus be viewed as a restriction of the more general vertex colouring problem to this special class of graphs. As we shall see, this relationship between the two types of colouring problem is reflected by a marked difference in our knowledge about their solutions: while there are only very rough estimates for χ, its sister χ' always takes one of two values, either Δ or $\Delta + 1$.

5.1 Colouring maps and planar graphs

If any result in graph theory has a claim to be known to the world outside, it is the following *four colour theorem* (which implies that every map can be coloured with at most four colours):

Theorem 5.1.1. (Four Colour Theorem)
Every planar graph is 4-colourable.

Some remarks about the proof of the four colour theorem and its history can be found in the notes at the end of this chapter. Here, we prove the following weakening:

Proposition 5.1.2. (Five Colour Theorem)
Every planar graph is 5-colourable.

(4.1.1)
(4.2.10)

n, m

Proof. Let G be a plane graph with $n \geqslant 6$ vertices and m edges. We assume inductively that every plane graph with fewer than n vertices can be 5-coloured. By Corollary 4.2.10,

$$d(G) = 2m/n \leqslant 2\,(3n-6)/n < 6\,;$$

v

H

c

let $v \in G$ be a vertex of degree at most 5. By the induction hypothesis, the graph $H := G - v$ has a vertex colouring $c\colon V(H) \to \{1, \dots, 5\}$. If c uses at most 4 colours for the neighbours of v, we can extend it to a 5-colouring of G. Let us assume, therefore, that v has exactly 5 neighbours, and that these have distinct colours.

D
$s_1, \dots, s_5$
$v_1, \dots, v_5$

Let D be an open disc around v, so small that it meets only those five straight edge segments of G that contain v. Let us enumerate these segments according to their cyclic position in D as $s_1, \dots, s_5$, and let vv_i be the edge containing s_i ($i = 1, \dots, 5$; Fig. 5.1.1). Without loss of generality we may assume that $c(v_i) = i$ for each i.

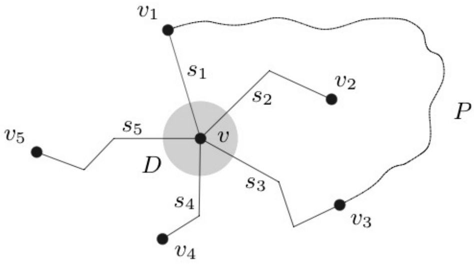

Fig. 5.1.1. The proof of the five colour theorem

P
C

Let us show first that every v_1–v_3 path $P \subseteq H$ separates v_2 from v_4 in H. Clearly, this is the case if and only if the cycle $C := vv_1Pv_3v$ separates v_2 from v_4 in G. We prove this by showing that v_2 and v_4 lie in different faces of C.

Let us pick an inner point x_2 of s_2 in D and an inner point x_4 of s_4 in D. Then in $D \setminus (s_1 \cup s_3) \subseteq \mathbb{R}^2 \setminus C$ every point can be linked by a polygonal arc to x_2 or to x_4. This implies that x_2 and x_4 (and hence also v_2 and v_4) lie in different faces of C: otherwise D would meet only one of the two faces of C, which would contradict the fact that v lies on the frontier of both these faces (Theorem 4.1.1).

$H_{i,j}$

Given $i, j \in \{1, \dots, 5\}$, let $H_{i,j}$ be the subgraph of H induced by the vertices coloured i or j. We may assume that the component C_1 of $H_{1,3}$ containing v_1 also contains v_3. Indeed, if we interchange the colours 1 and 3 at all the vertices of C_1, we obtain another 5-colouring of H; if $v_3 \notin C_1$, then v_1 and v_3 are both coloured 3 in this new colouring, and we may assign colour 1 to v. Thus, $H_{1,3}$ contains a v_1–v_3 path P. As shown above, P separates v_2 from v_4 in H. Since $P \cap H_{2,4} = \emptyset$, this means that v_2 and v_4 lie in different components of $H_{2,4}$. In the component containing v_2, we now interchange the colours 2 and 4, thus recolouring v_2 with colour 4. Now v no longer has a neighbour coloured 2, and we may give it this colour. □

As a backdrop to the two famous theorems above, let us cite another well-known result:

Theorem 5.1.3. (Grötzsch 1959)
Every planar graph not containing a triangle is 3-colourable.

5.2 Colouring vertices

How do we determine the chromatic number of a given graph? How can we *find* a vertex-colouring with as few colours as possible? How does the chromatic number relate to other graph invariants, such as average degree, connectivity or girth?

Straight from the definition of the chromatic number we may derive the following upper bound:

Proposition 5.2.1. *Every graph G with m edges satisfies*

$$\chi(G) \leqslant \tfrac{1}{2} + \sqrt{2m + \tfrac{1}{4}}\,.$$

Proof. Let c be a vertex colouring of G with $k = \chi(G)$ colours. Then G has at least one edge between any two colour classes: if not, we could have used the same colour for both classes. Thus, $m \geqslant \frac{1}{2}k(k-1)$. Solving this inequality for k, we obtain the assertion claimed. □

greedy algorithm

One obvious way to colour a graph G with not too many colours is the following *greedy algorithm*: starting from a fixed vertex enumeration $v_1, \dots, v_n$ of G, we consider the vertices in turn and colour each v_i with the first available colour—e.g., with the smallest positive integer not already used to colour any neighbour of v_i among $v_1, \dots, v_{i-1}$. In this way, we never use more than $\Delta(G) + 1$ colours, even for unfavourable choices of the enumeration $v_1, \dots, v_n$. If G is complete or an odd cycle, then this is even best possible.

In general, though, this upper bound of $\Delta + 1$ is rather generous, even for greedy colourings. Indeed, when we come to colour the vertex v_i in the above algorithm, we only need a supply of $d_{G[v_1,\dots,v_i]}(v_i) + 1$ rather than $d_G(v_i) + 1$ colours to proceed; recall that, at this stage, the algorithm ignores any neighbours v_j of v_i with $j > i$. Hence in most graphs, there will be scope for an improvement of the $\Delta + 1$ bound by choosing a particularly suitable vertex ordering to start with: one that picks vertices of large degree early (when most neighbours are ignored) and vertices of small degree last. Locally, the number $d_{G[v_1,\dots,v_i]}(v_i) + 1$ of colours required will be smallest if v_i has minimum degree in $G[v_1, \dots, v_i]$. But this is easily achieved: we just choose v_n first, with $d(v_n) = \delta(G)$, then choose as v_{n-1} a vertex of minimum degree in $G - v_n$, and so on.

colouring number $\mathrm{col}(G)$

The least number k such that G has a vertex enumeration in which each vertex is preceded by fewer than k of its neighbours is called the *colouring number* $\mathrm{col}(G)$ of G. The enumeration we just discussed shows that $\mathrm{col}(G) \leqslant \max_{H \subseteq G} \delta(H) + 1$. But for $H \subseteq G$ clearly also $\mathrm{col}(G) \geqslant \mathrm{col}(H)$ and $\mathrm{col}(H) \geqslant \delta(H) + 1$, since the 'back-degree' of the last vertex in any enumeration of H is just its ordinary degree in H, which is at least $\delta(H)$. So we have proved the following:

Proposition 5.2.2. *Every graph G satisfies*

$$\chi(G) \leqslant \operatorname{col}(G) = \max\{\,\delta(H) \mid H \subseteq G\,\} + 1\,.$$

□

[7.3]
[9.2.1]
[9.2.3]
[11.2.3]

Corollary 5.2.3. *Every graph G has a subgraph of minimum degree at least $\chi(G) - 1$.* □

The colouring number of a graph is closely related to its arboricity; see the remark following Theorem 2.4.4.

As we have seen, every graph G satisfies $\chi(G) \leqslant \Delta(G) + 1$, with equality for complete graphs and odd cycles. In all other cases, this general bound can be improved a little:

Theorem 5.2.4. (Brooks 1941)
Let G be a connected graph. If G is neither complete nor an odd cycle, then

$$\chi(G) \leqslant \Delta(G)\,.$$

Proof. We apply induction on $|G|$. If $\Delta(G) \leqslant 2$, then G is a path or a cycle, and the assertion is trivial. We therefore assume that $\Delta := \Delta(G) \geqslant 3$, and that the assertion holds for graphs of smaller order. Suppose that $\chi(G) > \Delta$.

Δ

Let $v \in G$ be a vertex and $H := G - v$. Then $\chi(H) \leqslant \Delta$: by induction, every component H' of H satisfies $\chi(H') \leqslant \Delta(H') \leqslant \Delta$ unless H' is complete or an odd cycle, in which case $\chi(H') = \Delta(H') + 1 \leqslant \Delta$ as every vertex of H' has maximum degree in H' and one such vertex is also adjacent to v in G.

v, H

Since H can be Δ-coloured but G cannot, we have the following:

> *Every Δ-colouring of H uses all the colours $1, \dots, \Delta$ on the neighbours of v; in particular, $d(v) = \Delta$.* (1)

Given any Δ-colouring of H, let us denote the neighbour of v coloured i by v_i, $i = 1, \dots, \Delta$. For all $i \neq j$, let $H_{i,j}$ denote the subgraph of H spanned by all the vertices coloured i or j.

$v_1, \dots, v_\Delta$
$H_{i,j}$

> *For all $i \neq j$, the vertices v_i and v_j lie in a common component $C_{i,j}$ of $H_{i,j}$.* (2)

$C_{i,j}$

Otherwise we could interchange the colours i and j in one of those components; then v_i and v_j would be coloured the same, contrary to (1).

> *$C_{i,j}$ is always a v_i–v_j path.* (3)

Indeed, let P be a v_i–v_j path in $C_{i,j}$. As $d_H(v_i) \leqslant \Delta - 1$, the neighbours of v_i have pairwise different colours: otherwise we could recolour v_i, contrary to (1). Hence the neighbour of v_i on P is its only neighbour in $C_{i,j}$, and similarly for v_j. Thus if $C_{i,j} \neq P$, then P has an inner vertex with three identically coloured neighbours in H; let u be the first such vertex on P (Fig. 5.2.1). Since at most $\Delta - 2$ colours are used on the neighbours of u, we may recolour u. But this makes $P\mathring{u}$ into a component of $H_{i,j}$, contradicting (2).

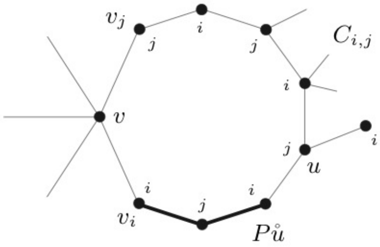

Fig. 5.2.1. The proof of (3) in Brooks's theorem

For distinct i, j, k, the paths $C_{i,j}$ and $C_{i,k}$ meet only in v_i. (4)

For if $v_i \neq u \in C_{i,j} \cap C_{i,k}$, then u has two neighbours coloured j and two coloured k, so we may recolour u. In the new colouring, v_i and v_j lie in different components of $H_{i,j}$, contrary to (2).

The proof of the theorem now follows easily. If the neighbours of v are pairwise adjacent, then each has Δ neighbours in $N(v) \cup \{v\}$ already, so $G = G[N(v) \cup \{v\}] = K^{\Delta+1}$. As G is complete, there is nothing to
$v_1, \dots, v_\Delta$ show. We may thus assume that $v_1 v_2 \notin G$, where $v_1, \dots, v_\Delta$ derive their
c names from some fixed Δ-colouring c of H. Let $u \neq v_2$ be the neighbour
u of v_1 on the path $C_{1,2}$; then $c(u) = 2$. Interchanging the colours 1 and 3
c' in $C_{1,3}$, we obtain a new colouring c' of H; let v'_i, $H'_{i,j}$, $C'_{i,j}$ etc. be defined with respect to c' in the obvious way. As a neighbour of $v_1 = v'_3$, our vertex u now lies in $C'_{2,3}$, since $c'(u) = c(u) = 2$. By (4) for c, however, the path $\mathring{v}_1 C_{1,2}$ retained its original colouring, so $u \in \mathring{v}_1 C_{1,2} \subseteq C'_{1,2}$. Hence $u \in C'_{2,3} \cap C'_{1,2}$, contradicting (4) for c'. □

As we have seen, a graph G of large chromatic number must have large maximum degree and colouring number, both at least $\chi(G) - 1$. What else does large $\chi(G)$ imply about G, in terms of other invariants or in structural terms? Are there, as in Kuratowski's theorem, some 'canonical' highly chromatic types of subgraph that must occur in every graph of large enough chromatic number?

One obvious possible cause for larger chromatic number, $\chi(G) \geqslant k$ say, is the presence of a K^k subgraph. This is a local property of G, compatible with arbitrary values of global invariants such as ε and κ. Hence, an assumption of $\chi(G) \geqslant k$ does not tell us anything about those invariants for G itself. It does, however, imply the existence of a subgraph where those invariants are large: by Corollary 5.2.3, G has a subgraph H with $\delta(H) \geqslant k-1$, and hence by Theorem 1.4.3, also a subgraph H' with $\kappa(H') \geqslant \lceil \frac{1}{4}k \rceil$.

But the presence of just any such subgraph is not equivalent to $\chi(G)$ being large, not even in a weak qualitative sense: as complete bipartite graphs show, no assumption of high[1] values of δ or κ alone can force χ to exceed 2, let alone to get arbitrarily large.

In particular, the collection of graphs of minimum degree at least $k-1$ or connectivity at least $\lceil \frac{1}{4}k \rceil$ cannot, as a whole, play the role of an easily identifiable Kuratowski-type set of minimal k-chromatic graphs. It may have a subclass that can. But no such set can be finite. Indeed, the following fundamental theorem of Erdős implies that for no k does there exist a finite set $\mathcal{H}$ of graphs of chromatic number at least 3 such that every graph of chromatic number at least k has a subgraph in $\mathcal{H}$:

Theorem 5.2.5. (Erdős 1959) [9.2.3]
For every integer k there exists a graph G with girth $g(G) > k$ and chromatic number $\chi(G) > k$.

Theorem 5.2.5 was first proved non-constructively using random graphs, and we shall give this proof in Chapter 11.2. Constructing graphs of large chromatic number and girth directly is not easy; cf. Exercise 24 for the simplest case.

The message of Erdős's theorem is that, contrary to our initial guess, large chromatic number can occur as a purely global phenomenon: note that locally, around each vertex, a graph of large girth looks just like a tree, and in particular is 2-colourable there. But what exactly can cause high chromaticity as a global phenomenon remains a mystery.

Nevertheless, there exists a simple—though not always short—procedure to construct all the graphs of chromatic number at least k. For each $k \in \mathbb{N}$, let us define the class of *k-constructible* graphs recursively as follows: *k-constructible*

(i) K^k is k-constructible.

(ii) If G is k-constructible and two vertices x, y of G are non-adjacent, then also $(G+xy)/xy$ is k-constructible.

(iii) If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible (Fig. 5.2.2).

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[1] High in absolute terms. In Chapter 7 we shall study the effect of edge densities that let ε get large also relative to the order of the graph. That is a much stronger assumption.

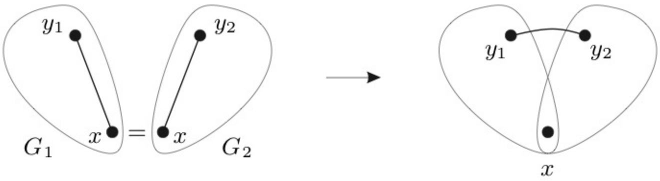

Fig. 5.2.2. The *Hajós construction* (iii)

One easily checks inductively that all k-constructible graphs—and hence their supergraphs—are at least k-chromatic. For example, any colouring of the graph $(G+xy)/xy$ in (ii) induces a colouring of G, and hence by inductive assumption uses at least k colours. Similarly, in any colouring of the graph constructed in (iii) the vertices y_1 and y_2 do not both have the same colour as x, so this colouring induces a colouring of either G_1 or G_2 and hence uses at least k colours.

It is remarkable, though, that the converse holds too:

Theorem 5.2.6. (Hajós 1961)
Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geqslant k$ if and only if G has a k-constructible subgraph.

Proof. Let G be a graph with $\chi(G) \geqslant k$; we show that G has a k-constructible subgraph. Suppose not; then $k \geqslant 3$. Adding some edges if necessary, let us make G edge-maximal with the property that none of its subgraphs is k-constructible. Now G is not a complete r-partite graph for any r: for then $\chi(G) \geqslant k$ would imply $r \geqslant k$, and G would contain the k-constructible graph K^k.

x, y_1, y_2
H_1, H_2

Since G is not a complete multipartite graph, non-adjacency is not an equivalence relation on $V(G)$. So there are vertices y_1, x, y_2 such that $y_1x, xy_2 \notin E(G)$ but $y_1y_2 \in E(G)$. Since G is edge-maximal without a k-constructible subgraph, each edge xy_i lies in some k-constructible subgraph H_i of $G + xy_i$ $(i = 1, 2)$.

H'_2
v' *etc.*

Let H'_2 be an isomorphic copy of H_2 that contains x and $H_2 - H_1$ but is otherwise disjoint from G, together with an isomorphism $v \mapsto v'$ from H_2 to H'_2 that fixes $H_2 \cap H'_2$ pointwise. Then $H_1 \cap H'_2 = \{x\}$, so

$$H := (H_1 \cup H'_2) - xy_1 - xy'_2 + y_1y'_2$$

is k-constructible by (iii). One vertex at a time, let us identify in H each vertex $v' \in H'_2 - G$ with its partner v; since vv' is never an edge of H, each of these identifications amounts to a construction step of type (ii). Eventually, we obtain the graph

$$(H_1 \cup H_2) - xy_1 - xy_2 + y_1y_2 \subseteq G\,;$$

this is the desired k-constructible subgraph of G. □

Does Hajós's theorem solve our Kuratowski-type problem for highly chromatic graphs, which was to find a class of graphs of chromatic number at least k with the property that every such graph has a subgraph in this class? Formally, it does—albeit with an infinite characterizing set (the set of k-constructible graphs). Unlike Kuratowski's characterization of planar graphs, however, this does not—at least not obviously—make Hajós's theorem a 'good characterization' of the graphs of chromatic number $< k$, in the sense of complexity theory. See the notes for details.

5.3 Colouring edges

Clearly, every graph G satisfies $\chi'(G) \geqslant \Delta(G)$. For bipartite graphs, we have equality here:

Proposition 5.3.1. (König 1916) [5.4.5]
Every bipartite graph G satisfies $\chi'(G) = \Delta(G)$.

Proof. We apply induction on $\|G\|$. For $\|G\| = 0$ the assertion holds. (1.6.1)
Now assume that $\|G\| \geqslant 1$, and that the assertion holds for graphs with
fewer edges. Let $\Delta := \Delta(G)$, pick an edge $xy \in G$, and choose a Δ- Δ, xy
edge-colouring of $G - xy$ by the induction hypothesis. Let us refer to
the edges coloured α as *α-edges*, etc. α-edge

In $G - xy$, each of x and y is incident with at most $\Delta - 1$ edges.
Hence there are $\alpha, \beta \in \{1, \dots, \Delta\}$ such that x is not incident with an α, β
α-edge and y is not incident with a β-edge. If $\alpha = \beta$, we can colour the edge xy with this colour and are done; so we may assume that $\alpha \neq \beta$, and that x is incident with a β-edge.

Let us extend this edge to a maximal walk W from x whose edges are coloured β and α alternately. Since no such walk contains a vertex twice (why not?), W exists and is a path. Moreover, W does not contain y: if it did, it would end in y on an α-edge (by the choice of β) and thus have even length, so $W + xy$ would be an odd cycle in G (cf. Proposition 1.6.1). We now recolour all the edges on W, swapping α with β. By the choice of α and the maximality of W, adjacent edges of $G - xy$ are still coloured differently. We have thus found a Δ-edge-colouring of $G - xy$ in which neither x nor y is incident with a β-edge. Colouring xy with β, we extend this colouring to a Δ-edge-colouring of G. □

Theorem 5.3.2. (Vizing 1964)
Every graph G satisfies

$$\Delta(G) \leqslant \chi'(G) \leqslant \Delta(G) + 1\,.$$

Proof. We prove the second inequality by induction on $\|G\|$. For $\|G\| = 0$ V, E
it is trivial. For the induction step let $G = (V, E)$ with $\Delta := \Delta(G) > 0$ be Δ

colouring

given, and assume that the assertion holds for graphs with fewer edges. Instead of '$(\Delta+1)$-edge-colouring' let us just say 'colouring'.

missing

α/β - path

For every edge $e \in G$ there exists a colouring of $G - e$, by the induction hypothesis. In such a colouring, the edges at a given vertex v use at most $d(v) \leqslant \Delta$ colours, so some colour $\beta \in \{1, \dots, \Delta+1\}$ is *missing at* v. For any other colour α, there is a unique maximal walk (possibly trivial) starting at v, whose edges are coloured alternately α and β. This walk is a path; we call it the *α/β - path from* v.

Suppose that G has no colouring. Then the following holds:

> *Given $xy \in E$, and any colouring of $G - xy$ in which the colour α is missing at x and the colour β is missing at y, the α/β - path from y ends in x.* (1)

Otherwise we could interchange the colours α and β along this path and colour xy with α, obtaining a colouring of G (contradiction).

α

G_i

Let $xy_0 \in G$ be an edge. By induction, $G_0 := G - xy_0$ has a colouring c_0. Let α be a colour missing at x in this colouring. Further, let $y_0, \dots, y_k$ be a maximal sequence of distinct neighbours of x in G such that $c_0(xy_{i+1})$ is missing in c_0 at y_i for every $i < k$. For each of the graphs $G_i := G - xy_i$ we define a colouring c_i, setting

c_i

$$c_i(e) := \begin{cases} c_0(xy_{j+1}) & \text{for } e = xy_j \text{ with } j \in \{0, \dots, i-1\} \\ c_0(e) & \text{otherwise;} \end{cases}$$

note that in each of these colourings the same colours are missing at x as in c_0.

β

Now let β be a colour missing at y_k in c_0. By (1), the α/β - path P from y_k in G_k (with respect to c_k) ends in x, with an edge yx coloured β since α is missing at x. Since y cannot serve as y_{k+1}, by the maximality of the sequence $y_0, \dots, y_k$, we thus have $y = y_i$ for some $0 \leqslant i < k$ (Fig. 5.3.1). By definition of c_k, therefore, $\beta = c_k(xy_i) = c_0(xy_{i+1})$. By

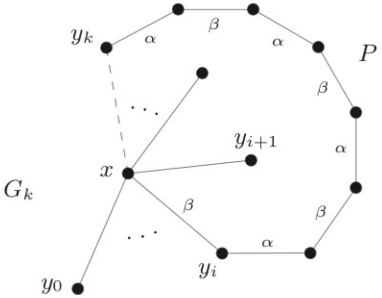

Fig. 5.3.1. The α/β - path P in $G_k = G - xy_k$

the choice of y_{i+1} this means that β was missing at y_i in c_0, and hence also in c_i. Now the α/β-path P' from y_i in G_i starts with $y_i P y_k$, since the edges of $P\mathring{x}$ are coloured the same in c_i as in c_k. But in c_0, and hence in c_i, there is no edge at y_k coloured β. Therefore P' ends in y_k, contradicting (1). $\square$

Vizing's theorem divides the finite graphs into two classes according to their chromatic index; graphs satisfying $\chi' = \Delta$ are called (imaginatively) *class 1*, those with $\chi' = \Delta + 1$ are *class 2*.

5.4 List colouring

In this section, we take a look at a relatively recent generalization of the concepts of colouring studied so far. This generalization may seem a little far-fetched at first glance, but it turns out to supply a fundamental link between the classical (vertex and edge) chromatic numbers of a graph and its other invariants.

Suppose we are given a graph $G = (V, E)$, and for each vertex of G a list of colours permitted at that particular vertex: when can we colour G (in the usual sense) so that each vertex receives a colour from its list? More formally, let $(S_v)_{v \in V}$ be a family of sets. We call a vertex colouring c of G with $c(v) \in S_v$ for all $v \in V$ a colouring *from the lists* S_v. The graph G is called *k-list-colourable*, or *k-choosable*, if, for every family $(S_v)_{v \in V}$ with $|S_v| = k$ for all v, there is a vertex colouring of G from the lists S_v. The least integer k for which G is k-choosable is the *list-chromatic number*, or *choice number* $\mathrm{ch}(G)$ of G.

k-choosable

choice number $\mathrm{ch}(G)$

List-colourings of edges are defined analogously. The least integer k such that G has an edge colouring from any family of lists of size k is the *list-chromatic index* $\mathrm{ch}'(G)$ of G; formally, we just set $\mathrm{ch}'(G) := \mathrm{ch}(L(G))$, where $L(G)$ is the line graph of G.

$\mathrm{ch}'(G)$

In principle, showing that a given graph is k-choosable is more difficult than proving it to be k-colourable: the latter is just the special case of the former where all lists are equal to $\{1, \ldots, k\}$. Thus,

$$\mathrm{ch}(G) \geqslant \chi(G) \quad \text{and} \quad \mathrm{ch}'(G) \geqslant \chi'(G)$$

for all graphs G.

In spite of these inequalities, many of the known upper bounds for the chromatic number have turned out to be valid for the choice number, too. Examples for this phenomenon include Brooks's theorem and Proposition 5.2.2; in particular, graphs of large choice number still have subgraphs of large minimum degree. On the other hand, it is easy to construct graphs for which the two invariants are wide apart (Exercise 26).

Taken together, these two facts indicate a little how far those general upper bounds on the chromatic number may be from the truth.

The following theorem shows that, in terms of its relationship to other graph invariants, the choice number differs fundamentally from the chromatic number. As mentioned before, there are 2-chromatic graphs of arbitrarily large minimum degree, e.g. the graphs $K_{n,n}$. The choice number, however, will be forced up by large values of invariants like δ, ε or κ:

Theorem 5.4.1. (Alon 1993)
There exists a function $f\colon \mathbb{N} \to \mathbb{N}$ *such that, given any integer* k*, all graphs* G *with average degree* $d(G) \geqslant f(k)$ *satisfy* $\mathrm{ch}(G) \geqslant k$.

The proof of Theorem 5.4.1 uses probabilistic methods as introduced in Chapter 11.

Although statements of the form $\mathrm{ch}(G) \leqslant k$ are formally stronger than the corresponding statement of $\chi(G) \leqslant k$, they can be easier to prove. A pretty example is the list version of the five colour theorem: every planar graph is 5-choosable. The proof of this does not use the five colour theorem (or even Euler's formula, on which the proof of the five colour theorem is based). We thus reobtain the five colour theorem as a corollary, with a very different proof.

Theorem 5.4.2. (Thomassen 1994)
Every planar graph is 5-choosable.

(4.2.8) *Proof.* We shall prove the following assertion for all plane graphs G with at least 3 vertices:

> *Suppose that every inner face of* G *is bounded by a triangle and its outer face by a cycle* $C = v_1 \dots v_k v_1$. *Suppose further that* v_1 *has already been coloured with the colour 1, and* v_2 *has been coloured 2. Suppose finally that with every other vertex of* C *a list of at least 3 colours is associated, and with every vertex of* $G - C$ *a list of at least 5 colours. Then the colouring of* v_1 *and* v_2 *can be extended to a colouring of* G *from the given lists.* (∗)

Let us check first that (∗) implies the assertion of the theorem. Let any plane graph be given, together with a list of 5 colours for each vertex. Add edges to this graph until it is a maximal plane graph G. By Proposition 4.2.8, G is a plane triangulation; let $v_1 v_2 v_3 v_1$ be the boundary of its outer face. We now colour v_1 and v_2 (differently) from their lists, and extend this colouring by (∗) to a colouring of G from the lists given.

Let us now prove (∗), by induction on $|G|$. If $|G| = 3$, then $G = C$ and the assertion is trivial. Now let $|G| \geqslant 4$, and assume (∗) for smaller graphs. If C has a chord vw, then vw lies on two unique cycles $C_1, C_2 \subseteq C + vw$ with $v_1v_2 \in C_1$ and $v_1v_2 \notin C_2$. For $i = 1, 2$, let G_i denote the subgraph of G induced by the vertices lying on C_i or in its inner face (Fig. 5.4.1). Applying the induction hypothesis first to G_1 and then—with the colours now assigned to v and w—to G_2 yields the desired colouring of G.

vw

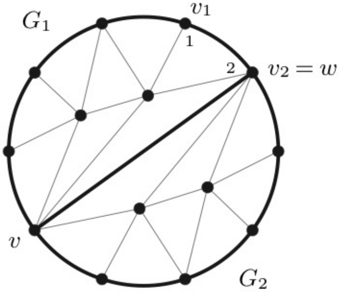

Fig. 5.4.1. The induction step with a chord vw; here the case of $w = v_2$

If C has no chord, let $v_1, u_1, \dots, u_m, v_{k-1}$ be the neighbours of v_k in their natural cyclic order order around v_k;[2] by definition of C, all those neighbours u_i lie in the inner face of C (Fig. 5.4.2). As the inner faces of C are bounded by triangles, $P := v_1u_1 \dots u_mv_{k-1}$ is a path in G, and $C' := P \cup (C - v_k)$ a cycle.

$u_1, \dots, u_m$

C'

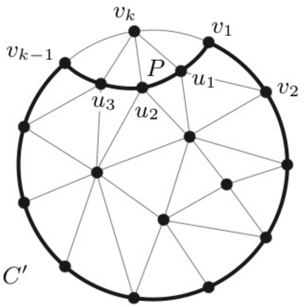

Fig. 5.4.2. The induction step without a chord

We now choose two different colours $j, \ell \neq 1$ from the list of v_k and delete these colours from the lists of all the vertices u_i. Then every list of a vertex on C' still has at least 3 colours, so by induction we may colour C' and its interior, i.e. the graph $G - v_k$. At least one of the two colours j, ℓ is not used for v_{k-1}, and we may assign that colour to v_k. □

---

[2] as in the first proof of the five colour theorem

As is often the case with induction proofs, the key to the proof above lies in its delicately balanced strengthening of the assertion proved. Compared with ordinary colouring, the task of finding a suitable strengthening is helped greatly by the possibility to give different vertices lists of different lengths, and thus to tailor the colouring problem more fittingly to the structure of the graph. This suggests that maybe in other unsolved colouring problems too it might be of advantage to aim straight for their list version, i.e. to prove an assertion of the form $\mathrm{ch}(G) \leqslant k$ instead of the formally weaker $\chi(G) \leqslant k$. Unfortunately, this approach fails for the four colour theorem: planar graphs are *not* in general 4-choosable.

As mentioned before, the chromatic number of a graph and its choice number may differ a lot. Surprisingly, however, no such examples are known for edge colourings. Indeed it has been conjectured that none exist:

List Colouring Conjecture. *Every graph G satisfies* $\mathrm{ch}'(G) = \chi'(G)$.

We shall prove the list colouring conjecture for bipartite graphs. As a tool we shall use orientations of graphs, defined in Chapter 1.10. If D is a directed graph and $v \in V(D)$, we denote by $N^+(v)$ the set, and by $d^+(v)$ the number, of vertices w such that D contains an edge directed from v to w.

$N^+(v)$

$d^+(v)$

To see how orientations come into play in the context of colouring, recall the greedy algorithm from Section 5.2. This colours the vertices of a graph G in turn, following a previously fixed ordering $(v_1, \ldots, v_n)$. This ordering defines an orientation of G if we orient every edge v_iv_j 'backwards', that is, from v_i to v_j if $i > j$. Then to determine a colour for v_i the algorithm only looks at previously coloured neighbours of v_i, those to which v_i sends a directed edge. In particular, if $d^+(v) < k$ for all vertices v, the algorithm will use at most k colours.

If we rewrite the proof of this fact (rather awkwardly) as a formal induction on k, we notice that the essential property of the set U of vertices coloured 1 is that every vertex in $G - U$ sends an edge to U: this ensures that $d^+_{G-U}(v) < d^+_G(v)$ for all $v \in G - U$, so we can colour $G - U$ with the remaining $k - 1$ colours by the induction hypothesis.

The following lemma generalizes these observations to list colouring, and to orientations D of G that do not necessarily come from a vertex enumeration but may contain some directed cycles. Let us call an independent set $U \subseteq V(D)$ a *kernel* of D if, for every vertex $v \in D - U$, there is an edge in D directed from v to a vertex in U. Note that kernels of non-empty directed graphs are themselves non-empty.

kernel

Lemma 5.4.3. *Let H be a graph and $(S_v)_{v \in V(H)}$ a family of lists. If H has an orientation D with $d^+(v) < |S_v|$ for every v, and such that every* *induced subgraph of D has a kernel, then H can be coloured from the lists S_v.*

Proof. We apply induction on $|H|$. For $|H| = 0$ we take the empty colouring. For the induction step, let $|H| > 0$. Let α be a colour occurring in one of the lists S_v, and let D be an orientation of H as stated. The vertices v with $\alpha \in S_v$ span a non-empty subgraph D' in D; by assumption, D' has a kernel $U \neq \emptyset$.

α D' U

Let us colour the vertices in U with α, and remove α from the lists of all the other vertices of D'. Since each of those vertices sends an edge to U, the modified lists S'_v for $v \in D - U$ again satisfy the condition $d^+(v) < |S'_v|$ in $D - U$. Since $D - U$ is an orientation of $H - U$, we can thus colour $H - U$ from those lists by the induction hypothesis. As none of these lists contains α, this extends our colouring $U \to \{\alpha\}$ to the desired list colouring of H. $\square$

In our proof of the list colouring conjecture for bipartite graphs we shall apply Lemma 5.4.3 only to colourings from lists of uniform length k. However, note that keeping list lengths variable is essential for the proof of the lemma itself: its simple induction could not be performed with uniform list lengths.

Theorem 5.4.4. (Galvin 1995)
Every bipartite graph G satisfies $\mathrm{ch}'(G) = \chi'(G)$.

Proof. Let $G =: (X \cup Y, E)$, where $\{X, Y\}$ is a vertex bipartition of G. Let us say that two edges of G *meet in X* if they share an end in X, and correspondingly for Y. Let $\chi'(G) =: k$, and let c be a k-edge-colouring of G.

(2.1.4) $X, Y,\ E$ k c

Clearly, $\mathrm{ch}'(G) \geqslant k$; we prove that $\mathrm{ch}'(G) \leqslant k$. Our plan is to use Lemma 5.4.3 to show that the line graph H of G is k-choosable. To apply the lemma, it suffices to find an orientation D of H with $d^+(e) < k$ for every vertex e of H, and such that every induced subgraph of D has a kernel. To define D, consider adjacent $e, e' \in E$, say with $c(e) < c(e')$. If e and e' meet in X, we orient the edge $ee' \in H$ from e' towards e; if e and e' meet in Y, we orient it from e to e' (Fig 5.4.3).

H D

Let us compute $d^+(e)$ for given $e \in E = V(D)$. If $c(e) = i$, say, then every $e' \in N^+(e)$ meeting e in X has its colour in $\{1, \dots, i-1\}$, and every $e' \in N^+(e)$ meeting e in Y has its colour in $\{i+1, \dots, k\}$. As any two neighbours e' of e meeting e either both in X or both in Y are themselves adjacent and hence coloured differently, this implies $d^+(e) < k$ as desired.

It remains to show that every induced subgraph D' of D has a kernel. This, however, is immediate by the stable marriage theorem (2.1.4) for G, if we interpret the directions in D as expressing preference. Indeed, given a vertex $v \in X \cup Y$ and edges $e, e' \in V(D')$ at v, write $e <_v e'$ if the edge

D'

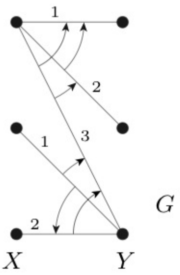

Fig. 5.4.3. Orienting the line graph of G

ee' of H is directed from e to e' in D. Then any stable matching in the graph $(X \cup Y, V(D'))$ for this set of preferences is a kernel in D'. □

(5.3.1) By Proposition 5.3.1, we now know the exact list-chromatic index of bipartite graphs:

Corollary 5.4.5. *Every bipartite graph* G *satisfies* $\mathrm{ch}'(G) = \Delta(G)$. □

5.5 Perfect graphs

As discussed in Section 5.2, a high chromatic number may occur as a purely global phenomenon: even when a graph has large girth, and thus locally looks like a tree, its chromatic number may be arbitrarily high. Since such 'global dependence' is obviously difficult to deal with, one may become interested in graphs where this phenomenon does not occur, i.e. whose chromatic number is high only when there is a local reason for it.

Before we make this precise, let us note two definitions for a graph G. $\omega(G)$ The greatest integer r such that $K^r \subseteq G$ is the *clique number* $\omega(G)$ of G, and the greatest integer r such that $\overline{K^r} \subseteq G$ (induced) is the *independence number* $\alpha(G)$ of G. Clearly, $\alpha(G) = \omega(\overline{G})$ and $\omega(G) = \alpha(\overline{G})$. $\alpha(G)$

perfect A graph is called *perfect* if every induced subgraph $H \subseteq G$ has chromatic number $\chi(H) = \omega(H)$, i.e. if the trivial lower bound of $\omega(H)$ colours always suffices to colour the vertices of H. Thus, while proving an assertion of the form $\chi(G) > k$ may in general be difficult, even in principle, for a given graph G, it can always be done for a perfect graph simply by exhibiting some K^{k+1} subgraph as a 'certificate' for non-colourability with k colours.

At first glance, the structure of the class of perfect graphs appears somewhat contrived: although it is closed under induced subgraphs (if only by explicit definition), it is not closed under taking general subgraphs or supergraphs, let alone minors (examples?). However, perfection is an important notion in graph theory: the fact that several fundamental classes of graphs are perfect (as if by fluke) may serve as a superficial indication of this.[3]

What graphs, then, are perfect? Bipartite graphs are, for instance. Less trivially, the complements of bipartite graphs are perfect, too—a fact equivalent to König's duality theorem 2.1.1 (Exercise 37). The so-called *comparability graphs* are perfect, and so are the *interval graphs* (see the exercises); both these turn up in numerous applications.

In order to study at least one such example in some detail, we prove here that the chordal graphs are perfect: a graph is *chordal* (or *triangulated*) if each of its cycles of length at least 4 has a chord, i.e. if it contains no induced cycles other than triangles.

chordal

To show that chordal graphs are perfect, we shall first characterize their structure. If G is a graph with induced subgraphs G_1, G_2 and S, such that $G = G_1 \cup G_2$ and $S = G_1 \cap G_2$, we say that G arises from G_1 and G_2 by *pasting* these graphs together along S.

pasting

Proposition 5.5.1. *A graph is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from complete graphs.*

[12.3.11]

Proof. If G is obtained from two chordal graphs G_1, G_2 by pasting them together along a complete subgraph, then G is clearly again chordal: any induced cycle in G lies in either G_1 or G_2, and is hence a triangle by assumption. Since complete graphs are chordal, this proves that all graphs constructible as stated are chordal.

Conversely, let G be a chordal graph. We show by induction on $|G|$ that G can be constructed as described. This is trivial if G is complete. We therefore assume that G is not complete, in particular that $|G| > 1$, and that all smaller chordal graphs are constructible as stated. Let $a, b \in G$ be two non-adjacent vertices, and let $X \subseteq V(G) \setminus \{a, b\}$ be a minimal a–b separator. Let C denote the component of $G - X$ containing a, and put $G_1 := G[V(C) \cup X]$ and $G_2 := G - C$. Then G arises from G_1 and G_2 by pasting these graphs together along $S := G[X]$.

a, b
X
C
G_1, G_2
S

Since G_1 and G_2 are both chordal (being induced subgraphs of G) and hence constructible by induction, it suffices to show that S is complete. Suppose, then, that $s, t \in S$ are non-adjacent. By the minimality of $X = V(S)$ as an a–b separator, both s and t have a neighbour in C. Hence, there is an X-path from s to t in G_1; we let P_1 be a shortest such

s, t

---

[3] The class of perfect graphs has duality properties with deep connections to optimization and complexity theory, which are far from understood. Theorem 5.5.6 shows the tip of an iceberg here; for more, the reader is referred to Lovász's survey cited in the notes.

path. Analogously, G_2 contains a shortest X-path P_2 from s to t. But then $P_1 \cup P_2$ is a chordless cycle of length $\geqslant 4$ (Fig. 5.5.1), contradicting our assumption that G is chordal. □

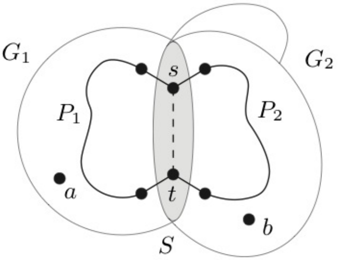

Fig. 5.5.1. If G_1 and G_2 are chordal, then so is G

Proposition 5.5.2. *Every chordal graph is perfect.*

Proof. Since complete graphs are perfect, it suffices by Proposition 5.5.1 to show that any graph G obtained from perfect graphs G_1, G_2 by pasting them together along a complete subgraph S is again perfect. So let $H \subseteq G$ be an induced subgraph; we show that $\chi(H) \leqslant \omega(H)$.

Let $H_i := H \cap G_i$ for $i = 1, 2$, and let $T := H \cap S$. Then T is again complete, and H arises from H_1 and H_2 by pasting along T. As an induced subgraph of G_i, each H_i can be coloured with $\omega(H_i)$ colours. Since T is complete and hence coloured injectively, two such colourings, one of H_1 and one of H_2, may be combined into a colouring of H with $\max\{\omega(H_1), \omega(H_2)\} \leqslant \omega(H)$ colours—if necessary by permuting the colours in one of the H_i. □

By definition, every induced subgraph of a perfect graph is again perfect. The property of perfection can therefore be characterized by forbidden induced subgraphs: there exists a set $\mathcal{H}$ of imperfect graphs such that any graph is perfect if and only if it has no induced subgraph isomorphic to an element of $\mathcal{H}$. (For example, we may choose as $\mathcal{H}$ the set of all imperfect graphs with vertices in $\mathbb{N}$.)

Naturally, one would like to keep $\mathcal{H}$ as small as possible. It is one of the deepest results in graph theory that $\mathcal{H}$ need only contain two types of graph: the odd cycles of length $\geqslant 5$ and their complements. (Neither of these are perfect; cf. Theorem 5.5.4 below.) This fact, the famous *strong perfect graph conjecture* of Berge (1963), was proved only very recently:

strong perfect graph theorem

Theorem 5.5.3. (Chudnovsky, Robertson, Seymour & Thomas 2006) *A graph G is perfect if and only if neither G nor $\overline{G}$ contains an odd cycle of length at least 5 as an induced subgraph.*

The proof of the strong perfect graph theorem is long and technical, and it would not be too illuminating to attempt to sketch it. To shed more light on the notion of perfection, we instead give two direct proofs of its most important consequence: the *perfect graph theorem*, formerly Berge's *weak perfect graph conjecture*:

perfect graph theorem

Theorem 5.5.4. (Lovász 1972)
A graph is perfect if and only if its complement is perfect.

The first proof we give for Theorem 5.5.4 is Lovász's original proof, which is still unsurpassed in its clarity and the amount of 'feel' for the problem it conveys. Our second proof, due to Gasparian (1996), is an elegant linear algebra proof of another theorem of Lovász's (Theorem 5.5.6), which easily implies Theorem 5.5.4.

expanding a vertex

Let us prepare our first proof of Theorem 5.5.4 by a lemma. Let G be a graph and $x \in G$ a vertex, and let G' be obtained from G by adding a vertex x' and joining it to x and all the neighbours of x. We say that G' is obtained from G by *expanding* the vertex x to an edge xx' (Fig. 5.5.2).

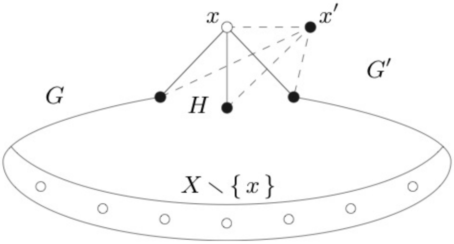

Fig. 5.5.2. Expanding the vertex x in the proof of Lemma 5.5.5

Lemma 5.5.5. *Any graph obtained from a perfect graph by expanding a vertex is again perfect.*

x, x'

Proof. We use induction on the order of the perfect graph considered. Expanding the vertex of K^1 yields K^2, which is perfect. For the induction step, let G be a non-trivial perfect graph, and let G' be obtained from G by expanding a vertex $x \in G$ to an edge xx'. For our proof that G' is perfect it suffices to show $\chi(G') \leqslant \omega(G')$: every proper induced subgraph H of G' is either isomorphic to an induced subgraph of G or obtained from a proper induced subgraph of G by expanding x; in either case, H is perfect by assumption and the induction hypothesis, and can hence be coloured with $\omega(H)$ colours.

ω

Let $\omega(G) =: \omega$; then $\omega(G') \in \{\omega, \omega+1\}$. If $\omega(G') = \omega+1$, then

$$\chi(G') \leqslant \chi(G)+1 = \omega+1 = \omega(G')$$

X H

and we are done. So let us assume that $\omega(G') = \omega$. Then x lies in no $K^\omega \subseteq G$: together with x', this would yield a $K^{\omega+1}$ in G'. Let us colour G with ω colours. Since every $K^\omega \subseteq G$ meets the colour class X of x but not x itself, the graph $H := G - (X \smallsetminus \{x\})$ has clique number $\omega(H) < \omega$ (Fig. 5.5.2). Since G is perfect, we may thus colour H with $\omega - 1$ colours. Now X is independent, so the set $(X \smallsetminus \{x\}) \cup \{x'\} = V(G' - H)$ is also independent. We can therefore extend our $(\omega-1)$-colouring of H to an ω-colouring of G', showing that $\chi(G') \leqslant \omega = \omega(G')$ as desired. □

$G = (V, E)$ $\mathcal{K}$ α $\mathcal{A}$

Proof of Theorem 5.5.4. Applying induction on $|G|$, we show that the complement $\overline{G}$ of any perfect graph $G = (V, E)$ is again perfect. For $|G| = 1$ this is trivial, so let $|G| \geqslant 2$ for the induction step. Let $\mathcal{K}$ denote the set of all vertex sets of complete subgraphs of G. Put $\alpha(G) =: \alpha$, and let $\mathcal{A}$ be the set of all independent vertex sets A in G with $|A| = \alpha$.

Every proper induced subgraph of $\overline{G}$ is the complement of a proper induced subgraph of G, and is hence perfect by induction. For the perfection of $\overline{G}$ it thus suffices to prove $\chi(\overline{G}) \leqslant \omega(\overline{G})$ $(= \alpha)$. To this end, we shall find a set $K \in \mathcal{K}$ such that $K \cap A \neq \emptyset$ for all $A \in \mathcal{A}$; then

$$\omega(\overline{G} - K) = \alpha(G - K) < \alpha = \omega(\overline{G}),$$

so by the induction hypothesis

$$\chi(\overline{G}) \leqslant \chi(\overline{G} - K) + 1 = \omega(\overline{G} - K) + 1 \leqslant \omega(\overline{G})$$

as desired.

A_K G_x

Suppose there is no such K; thus, for every $K \in \mathcal{K}$ there exists a set $A_K \in \mathcal{A}$ with $K \cap A_K = \emptyset$. Let us replace in G every vertex x by a complete graph G_x of order

$k(x)$

$$k(x) := \bigl|\{\, K \in \mathcal{K} \mid x \in A_K \,\}\bigr|,$$

G'

joining all the vertices of G_x to all the vertices of G_y whenever x and y are adjacent in G. The graph G' thus obtained has vertex set $\bigcup_{x \in V} V(G_x)$, and two vertices $v \in G_x$ and $w \in G_y$ are adjacent in G' if and only if $x = y$ or $xy \in E$. Moreover, G' can be obtained by repeated vertex expansion from the graph $G[\{\, x \in V \mid k(x) > 0 \,\}]$. Being an induced subgraph of G, this latter graph is perfect by assumption, so G' is perfect by Lemma 5.5.5. In particular,

$$\chi(G') \leqslant \omega(G'). \tag{1}$$

In order to obtain a contradiction to (1), we now compute in turn the actual values of $\omega(G')$ and $\chi(G')$. By construction of G', every maximal complete subgraph of G' has the form $G'[\bigcup_{x\in X} G_x]$ for some $X \in \mathcal{K}$. So there exists a set $X \in \mathcal{K}$ such that X

$$\begin{aligned}
\omega(G') &= \sum_{x\in X} k(x) \\
&= \big|\{\,(x,K) : x \in X,\ K \in \mathcal{K},\ x \in A_K\,\}\big| \\
&= \sum_{K\in\mathcal{K}} |X \cap A_K| \\
&\leqslant |\mathcal{K}| - 1\,; \qquad (2)
\end{aligned}$$

the last inequality follows from the fact that $|X \cap A_K| \leqslant 1$ for all K (since A_K is independent but $G[X]$ is complete), and $|X \cap A_X| = 0$ (by the choice of A_X). On the other hand,

$$\begin{aligned}
|G'| &= \sum_{x\in V} k(x) \\
&= \big|\{\,(x,K) : x \in V,\ K \in \mathcal{K},\ x \in A_K\,\}\big| \\
&= \sum_{K\in\mathcal{K}} |A_K| \\
&= |\mathcal{K}| \cdot \alpha\,.
\end{aligned}$$

As $\alpha(G') \leqslant \alpha$ by construction of G', this implies

$$\chi(G') \geqslant \frac{|G'|}{\alpha(G')} \geqslant \frac{|G'|}{\alpha} = |\mathcal{K}|\,. \qquad (3)$$

Putting (2) and (3) together we obtain

$$\chi(G') \geqslant |\mathcal{K}| > |\mathcal{K}| - 1 \geqslant \omega(G')\,,$$

a contradiction to (1). □

At first reading, the proof of Theorem 5.5.4 appears magical: it starts with an unmotivated lemma about expanding a vertex, shifts the problem to a strange graph G' obtained in this way, performs some double counting—and finished. With hindsight, however, we can understand it a little better. The proof is completely natural up to the point where we assume that for every $K \in \mathcal{K}$ there is an $A_K \in \mathcal{A}$ such that $K \cap A_K = \emptyset$. To show that this contradicts our assumption that G is perfect, we would like to show next that its subgraph $\tilde{G}$ induced by all the A_K has a chromatic number that is too large, larger than its clique number. And, as always when we try to bound the chromatic number from below, our only hope is to bound $|\tilde{G}|/\alpha$ instead, i.e. to show that this is larger than $\omega(\tilde{G})$.

But is the bound of $|\tilde{G}|/\alpha$ likely to reflect the true value of $\chi(\tilde{G})$? In one special case it is: if the sets A_K happen to be disjoint, we have $|\tilde{G}| = |\mathcal{K}| \cdot \alpha$ and $\chi(\tilde{G}) = |\mathcal{K}|$, with the A_K as colour classes. Of course, the sets A_K will not in general be disjoint. But we can make them so: by replacing every vertex x with $k(x)$ vertices, where $k(x)$ is the number of sets A_K it lives in! This is the idea behind G'. What remains is to endow G' with the right set of edges to make it perfect (assuming that G is perfect)—which leads straight to the definition of vertex expansion and Lemma 5.5.5.

Since the following characterization of perfection is symmetrical in G and $\overline{G}$, it clearly implies Theorem 5.5.4. As our proof of Theorem 5.5.6 will again be from first principles, we thus obtain a second and independent proof of Theorem 5.5.4.

Theorem 5.5.6. (Lovász 1972)
A graph G is perfect if and only if

$$|H| \leqslant \alpha(H) \cdot \omega(H) \tag{$*$}$$

for all induced subgraphs $H \subseteq G$.

v_i, n α, ω *Proof.* Let us write $V(G) =: \{v_1, \dots, v_n\}$, and put $\alpha := \alpha(G)$ and $\omega := \omega(G)$. The necessity of $(*)$ is immediate: if G is perfect, then every induced subgraph H of G can be partitioned into at most $\omega(H)$ colour classes each containing at most $\alpha(H)$ vertices, and $(*)$ follows.

To prove sufficiency, we apply induction on $n = |G|$. Assume that every induced subgraph H of G satisfies $(*)$, and suppose that G is not perfect. By the induction hypothesis, every proper induced subgraph of G is perfect. Hence, every non-empty independent set $U \subseteq V(G)$ satisfies

$$\chi(G - U) = \omega(G - U) = \omega\,. \tag{1}$$

Indeed, while the first equality is immediate from the perfection of $G - U$, the second is easy: '$\leqslant$' is obvious, while $\chi(G - U) < \omega$ would imply $\chi(G) \leqslant \omega$, so G would be perfect contrary to our assumption.

Let us apply (1) to a singleton $U = \{u\}$ and consider an ω-colouring of $G - u$. Let K be the vertex set of any K^ω in G. Clearly,

$$\text{if } u \notin K \text{ then } K \text{ meets every colour class of } G - u; \tag{2}$$

$$\text{if } u \in K \text{ then } K \text{ meets all but exactly one colour class of } G - u. \tag{3}$$

A_0 Let $A_0 = \{u_1, \dots, u_\alpha\}$ be an independent set in G of size α. Let $A_1, \dots, A_\omega$ be the colour classes of an ω-colouring of $G - u_1$, let

A_i K_i

$A_{\omega+1}, \dots, A_{2\omega}$ be the colour classes of an ω-colouring of $G - u_2$, and so on; altogether, this gives us $\alpha\omega + 1$ independent sets $A_0, A_1, \dots, A_{\alpha\omega}$ in G. For each $i = 0, \dots, \alpha\omega$, there exists by (1) a $K^\omega \subseteq G - A_i$; we denote its vertex set by K_i.

Note that if K is the vertex set of any K^ω in G, then

$$K \cap A_i = \emptyset \textit{ for exactly one } i \in \{0, \dots, \alpha\omega\}. \tag{4}$$

Indeed, if $K \cap A_0 = \emptyset$ then $K \cap A_i \neq \emptyset$ for all $i \neq 0$, by definition of A_i and (2). Similarly if $K \cap A_0 \neq \emptyset$, then $|K \cap A_0| = 1$, so $K \cap A_i = \emptyset$ for exactly one $i \neq 0$: apply (3) to the unique vertex $u \in K \cap A_0$, and (2) to all the other vertices $u \in A_0$.

J A B

Let J be the real $(\alpha\omega + 1) \times (\alpha\omega + 1)$ matrix with zero entries in the main diagonal and all other entries 1. Let $A = (a_{ij})$ be the real $(\alpha\omega + 1) \times n$ matrix whose rows are the incidence vectors of the sets A_i with $V(G)$: where $a_{ij} = 1$ if $v_j \in A_i$, and $a_{ij} = 0$ otherwise. Similarly, let B denote the real $n \times (\alpha\omega + 1)$ matrix whose columns are the incidence vectors of the sets K_j with $V(G)$. Now while $|A_i \cap K_i| = 0$ for all i by the choice of K_i, we have $A_i \cap K_j \neq \emptyset$ and hence $|A_i \cap K_j| = 1$ whenever $i \neq j$, by (4). Thus,

$$AB = J.$$

Since J is non-singular, this implies that A has rank $\alpha\omega + 1$. In particular, $n \geqslant \alpha\omega + 1$, which contradicts $(*)$ for $H := G$. □

Exercises

1.$^-$ Show that the four colour theorem does indeed solve the map colouring problem stated in the first sentence of the chapter. Conversely, does the 4-colourability of every map imply the four colour theorem?

2.$^-$ Show that, for the map colouring problem above, it suffices to consider maps such that no point lies on the boundary of more than three countries. How does this affect the proof of the four colour theorem?

3.$^-$ Try to turn the proof of the five colour theorem into one of the four colour theorem, as follows. Defining v and H as before, assume inductively that H has a 4-colouring; then proceed as before. Where does the proof fail?

4. Calculate the chromatic number of a graph in terms of the chromatic numbers of its blocks.

5.$^-$ Show that every graph G has a vertex ordering for which the greedy algorithm uses only $\chi(G)$ colours.

6. For every $n > 1$, find a bipartite graph on $2n$ vertices, ordered in such a way that the greedy algorithm uses n rather than 2 colours.

7. Consider the following approach to vertex colouring. First, find a maximal independent set of vertices and colour these with colour 1; then find a maximal independent set of vertices in the remaining graph and colour those 2, and so on. Compare this algorithm with the greedy algorithm: which is better?

8. Show that the bound of Proposition 5.2.2 is always at least as sharp as that of Proposition 5.2.1.

9. Find a lower bound for the colouring number in terms of average degree.

10. Find a function f such that every graph of arboricity at least $f(k)$ has colouring number at least k, and a function g such that every graph of colouring number at least $g(k)$ has arboricity at least k, for all $k \in \mathbb{N}$.

11.$^-$ A k-chromatic graph is called *critically k-chromatic*, or just *critical*, if $\chi(G - v) < k$ for every $v \in V(G)$. Show that every k-chromatic graph has a critical k-chromatic induced subgraph, and that any such subgraph has minimum degree at least $k - 1$.

12. Determine the critical 3-chromatic graphs.

13.$^+$ Show that every critical k-chromatic graph is $(k-1)$-edge-connected.

14. Given $k \in \mathbb{N}$, find a constant $c_k > 0$ such that every large enough graph G with $\alpha(G) \leqslant k$ contains a cycle of length at least $c_k\,|G|$.

15.$^-$ Find a graph G for which Brooks's theorem yields a significantly weaker bound on $\chi(G)$ than Proposition 5.2.2.

16.$^+$ Show that, in order to prove Brooks's theorem for a graph $G = (V, E)$, we may assume that $\kappa(G) \geqslant 2$ and $\Delta(G) \geqslant 3$. Prove the theorem under these assumptions, showing first the following two lemmas.

    (i) Let $v_1, \dots, v_n$ be an enumeration of V. If every v_i $(i < n)$ has a neighbour v_j with $j > i$, and if $v_1v_n, v_2v_n \in E$ but $v_1v_2 \notin E$, then the greedy algorithm uses at most $\Delta(G)$ colours.

    (ii) If G is not complete and v_n has maximum degree in G, then v_n has neighbours v_1, v_2 as in (i).

17.$^+$ Show that the following statements are equivalent for a graph G:

    (i) $\chi(G) \leqslant k$;

    (ii) G has an orientation without directed paths of length k;

    (iii) G has an acyclic such orientation (one without directed cycles).

18. Given a graph G and $k \in \mathbb{N}$, let $P_G(k)$ denote the number of vertex colourings $V(G) \to \{1, \dots, k\}$. Show that P_G is a polynomial in k of degree $n := |G|$, in which the coefficient of k^n is 1 and the coefficient of k^{n-1} is $-\|G\|$. (P_G is called the *chromatic polynomial* of G.)

    (Hint. Apply induction on $\|G\|$.)

19.$^+$ Determine the class of all graphs G for which $P_G(k) = k\,(k-1)^{n-1}$. (As in the previous exercise, let $n := |G|$, and let P_G denote the chromatic polynomial of G.)

20. In the definition of 'k-constructible', replace axioms (ii) and (iii) by

    (ii)′ *Every supergraph of a k-constructible graph is k-constructible.*

    (iii)′ *If x, y_1, y_2 are distinct vertices of a graph G and $y_1y_2 \in E(G)$, and if both $G + xy_1$ and $G + xy_2$ are k-constructible, then G is k-constructible.*

    Show that a graph is k-constructible with respect to this new definition if and only if its chromatic number is at least k.

21.$^-$ An $n \times n$-matrix with entries from $\{1, \ldots, n\}$ is called a *Latin square* if every element of $\{1, \ldots, n\}$ appears exactly once in each column and exactly once in each row. Recast the problem of constructing Latin squares as a colouring problem.

22. Without using Proposition 5.3.1, show that $\chi'(G) = k$ for every k-regular bipartite graph G.

23. Prove Proposition 5.3.1 from the statement of the previous exercise.

24.$^+$ For every $k \in \mathbb{N}$, construct a triangle-free k-chromatic graph.

25.$^-$ Without using Theorem 5.4.2, show that every plane graph is 6-list-colourable.

26. For every integer k, find a 2-chromatic graph whose choice number is at least k.

27.$^-$ Find a general upper bound for $\text{ch}'(G)$ in terms of $\chi'(G)$.

28. Compare the choice number of a graph with its colouring number: which is greater? Can you prove the analogue of Theorem 5.4.1 for the colouring number?

29.$^+$ Prove that the choice number of K_2^r is r.

30. The *total chromatic number* $\chi''(G)$ of a graph $G = (V, E)$ is the least number of colours needed to colour the vertices and edges of G simultaneously so that any adjacent or incident elements of $V \cup E$ are coloured differently. The *total colouring conjecture* says that $\chi''(G) \leqslant \Delta(G) + 2$. Bound the total chromatic number from above in terms of the list-chromatic index, and use this bound to deduce a weakening of the total colouring conjecture from the list colouring conjecture.

31.$^-$ Does every oriented graph have a kernel? If not, does every graph admit an orientation in which every induced subgraph has a kernel? If not, does every graph admit an orientation that has a kernel?

32.$^+$ Prove Richardson's theorem that every directed graph without odd directed cycles has a kernel.

33. Show that every bipartite planar graph is 3-list-colourable.

    (Hint. Apply the previous exercise and Lemma 5.4.3.)

34.$^-$ Show that perfection is closed neither under edge deletion nor under edge contraction.

35.$^-$ Deduce Theorem 5.5.6 from the strong perfect graph theorem.

36. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two sets of imperfect graphs, each minimal with the property that a graph is perfect if and only if it has no induced subgraph in $\mathcal{H}_i$ $(i = 1, 2)$. Do $\mathcal{H}_1$ and $\mathcal{H}_2$ contain the same graphs, up to isomorphism?

37. Use König's Theorem 2.1.1 to show that the complement of any bipartite graph is perfect.

38. Using the results of this chapter, find a one-line proof of the following theorem of König, the dual of Theorem 2.1.1: in any bipartite graph without isolated vertices, the minimum number of edges meeting all vertices equals the maximum number of independent vertices.

39. A graph is called a *comparability graph* if there exists a partial ordering of its vertex set such that two vertices are adjacent if and only if they are comparable. Show that every comparability graph is perfect.

40. A graph G is called an *interval graph* if there exists a set $\{ I_v \mid v \in V(G) \}$ of real intervals such that $I_u \cap I_v \neq \emptyset$ if and only if $uv \in E(G)$.

    (i) Show that every interval graph is chordal.

    (ii) Show that the complement of any interval graph is a comparability graph.

    (Conversely, a chordal graph is an interval graph if its complement is a comparability graph; this is a theorem of Gilmore and Hoffman (1964).)

41. Show that $\chi(H) \in \{\omega(H)\,,\, \omega(H)+1\}$ for every line graph H.

42.$^+$ Characterize the graphs whose line graphs are perfect.

43. Show that a graph G is perfect if and only if every non-empty induced subgraph H of G contains an independent set $A \subseteq V(H)$ such that $\omega(H - A) < \omega(H)$.

44.$^+$ Consider the graphs G for which every induced subgraph H has the property that every maximal complete subgraph of H meets every maximal independent vertex set in H.

    (i) Show that these graphs G are perfect.

    (ii) Show that these graphs G are precisely the graphs not containing an induced copy of P^3.

45.$^+$ Show that in every perfect graph G one can find a set $\mathcal{A}$ of independent vertex sets and a set $\mathcal{O}$ of vertex sets of complete subgraphs such that $\bigcup \mathcal{A} = V(G) = \bigcup \mathcal{O}$ and every set in $\mathcal{A}$ meets every set in $\mathcal{O}$.

    (Hint. Lemma 5.5.5.)

46.$^+$ Let G be a perfect graph. As in the proof of Theorem 5.5.4, replace every vertex x of G with a perfect graph G_x (not necessarily complete). Show that the resulting graph G' is again perfect.

Notes

The authoritative reference work on all questions of graph colouring is T.R. Jensen & B. Toft, *Graph Coloring Problems*, Wiley 1995. Starting with a brief survey of the most important results and areas of research in the field, this monograph gives a detailed account of over 200 open colouring problems, complete with extensive background surveys and references. Most of the remarks below are discussed comprehensively in this book, and all the references for this chapter can be found there.

The *four colour problem*, whether every map can be coloured with four colours so that adjacent countries are shown in different colours, was raised by a certain Francis Guthrie in 1852. He put the question to his brother Frederick, who was then a mathematics undergraduate in Cambridge. The problem was first brought to the attention of a wider public when Cayley presented it to the London Mathematical Society in 1878. A year later, Kempe published an incorrect proof, which was in 1890 modified by Heawood into a proof of the five colour theorem. In 1880, Tait announced 'further proofs' of the four colour conjecture, which never materialized; see the notes for Chapter 10.

The first generally accepted proof of the four colour theorem was published by Appel and Haken in 1977. The proof builds on ideas that can be traced back as far as Kempe's paper, and were developed largely by Birkhoff and Heesch. Very roughly, the proof sets out first to show that every plane triangulation must contain at least one of 1482 certain 'unavoidable configurations'. In a second step, a computer is used to show that each of those configurations is 'reducible', i.e., that any plane triangulation containing such a configuration can be 4-coloured by piecing together 4-colourings of smaller plane triangulations. Taken together, these two steps amount to an inductive proof that all plane triangulations, and hence all planar graphs, can be 4-coloured.

Appel & Haken's proof has not been immune to criticism, not only because of their use of a computer. The authors responded with a 741 page long algorithmic version of their proof, which addresses the various criticisms and corrects a number of errors (e.g. by adding more configurations to the 'unavoidable' list): K. Appel & W. Haken, *Every Planar Map is Four Colorable*, American Mathematical Society 1989. A much shorter proof, which is based on the same ideas (and, in particular, uses a computer in the same way) but can be more readily verified both in its verbal and its computer part, has been given by N. Robertson, D. Sanders, P.D. Seymour & R. Thomas, The four-colour theorem, *J. Comb. Theory B* **70** (1997), 2–44.

A relatively short proof of Grötzsch's theorem was found by C. Thomassen, A short list color proof of Grötzsch's theorem, *J. Comb. Theory B* **88** (2003), 189–192. Although not touched upon in this chapter, colouring problems for graphs embedded in surfaces other than the plane form a substantial and interesting part of colouring theory; see B. Mohar & C. Thomassen, *Graphs on Surfaces*, Johns Hopkins University Press 2001.

The proof of Brooks's theorem indicated in Exercise 16, where the greedy algorithm is applied to a carefully chosen vertex ordering, is due to Lovász (1973). Lovász (1968) was also the first to *construct* graphs of arbitrarily large girth and chromatic number, graphs whose existence Erdős had proved by probabilistic methods ten years earlier in Graph theory and probability, *Can. J. Math.* **11** (1959), 34–38. Another constructive proof can be found in J. Nešetřil & V. Rödl, Sparse Ramsey graphs, *Combinatorica* **4** (1984), 71–78.

A. Urquhart, The graph constructions of Hajós and Ore, *J. Graph Theory* **26** (1997), 211–215, showed that not only do the graphs of chromatic number at least k each *contain* a k-constructible graph (as by Hajós's theorem); they are in fact all themselves k-constructible. Note that, in the course of constructing a given graph, the order of the graphs constructed on the way can go both up and down, depending on which rule is applied at each step. This means that there is no obvious upper bound on the number of steps needed to construct a given graph, and indeed no such bound is known. In particular, Hajós's theorem does not provide bounded-length 'certificates' for the property of having chromatic number at least k. Unlike Kuratowski's theorem, it is therefore not a 'good characterization' in the sense of complexity theory. (See Chapter 12.5, the notes for Chapter 10, and the end of the notes for Chapter 12 for more details.)

Algebraic tools for showing that the chromatic number of a graph is large have been developed by Kleitman & Lovász (1982), by Alon & Tarsi (see Alon's paper cited below), and by Babson & Kozlov (2007).

List colourings were first introduced in 1976 by Vizing. Among other things, Vizing proved the list-colouring equivalent of Brooks's theorem. Voigt (1993) constructed a plane graph of order 238 that is not 4-choosable; thus, Thomassen's list version of the five colour theorem is best possible. A stimulating survey on the list-chromatic number and how it relates to the more classical graph invariants (including a proof of Theorem 5.4.1) is given by N. Alon, Restricted colorings of graphs, in (K. Walker, ed.) *Surveys in Combinatorics*, LMS Lecture Notes **187**, Cambridge University Press 1993. Both the list colouring conjecture and Galvin's proof of the bipartite case are originally stated for multigraphs. Kahn (1994) proved that the conjecture is asymptotically correct, as follows: given any $\epsilon > 0$, every graph G with large enough maximum degree satisfies $\mathrm{ch}'(G) \leqslant (1+\epsilon)\Delta(G)$.

The total colouring conjecture (Exercise 30) was proposed around 1965 by Vizing and by Behzad; see Jensen & Toft for details.

A gentle introduction to the basic facts about perfect graphs and their applications is given by M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press 1980. A more comprehensive treatment is given in A. Schrijver, *Combinatorial optimization*, Springer 2003. Surveys on various aspects of perfect graphs are included in *Perfect Graphs* by J. Ramirez-Alfonsin & B. Reed (eds.), Wiley 2001. Our first proof of the perfect graph theorem, Theorem 5.5.4, follows Lovász's survey on perfect graphs in (L.W. Beineke and R.J. Wilson, eds.) *Selected Topics in Graph Theory 2*, Academic Press 1983. Our second proof, the proof of Theorem 5.5.6, is due to G.S. Gasparian, Minimal imperfect graphs: a simple approach, *Combinatorica* **16** (1996), 209–212. Theorem 5.5.3 was proved by Chudnovsky, Robertson, Seymour and Thomas, The strong perfect graph theorem, *Ann. Math.* **164** (2006), 51–229. Chudnovsky, Cornuejols, Liu, Seymour and Vušković, Recognizing Berge graphs, *Combinatorica* **25** (2005), 143–186, constructed an $O(n^9)$ algorithm testing for odd 'holes' (induced odd cycles of length at least 5) and odd 'antiholes' (their induced complements), and thus by the theorem for perfection.