3 Connectivity

Our definition of k-connectedness, given in Chapter 1.4, is somewhat unintuitive. It does not tell us much about 'connections' in a k-connected graph: all it says is that we need at least k vertices to *dis*connect it. The following definition—which, incidentally, implies the one above—might have been more descriptive: 'a graph is *k-connected* if any two of its vertices can be joined by k independent paths'.

It is one of the classic results of graph theory that these two definitions are in fact equivalent, are dual aspects of the same property. We shall study this theorem of Menger (1927) in some depth in Section 3.3.

In Sections 3.1 and 3.2, we investigate the structure of the 2-connected and the 3-connected graphs. For these small values of k it is still possible to give a simple general description of how these graphs can be constructed.

In Sections 3.4 and 3.5 we look at other concepts of connectedness, more recent than the standard one but no less important: the number of H-paths in G for a subgraph H of G, and the existence of disjoint paths in G linking up specified pairs of vertices.

3.1 2-Connected graphs and subgraphs

The simplest 2-connected graphs are the cycles. All the others can be constructed inductively from a cycle by adding paths:

Proposition 3.1.1. *A graph is 2-connected if and only if it can be constructed from a cycle by successively adding H-paths to graphs H already constructed (Fig. 3.1.1).* [4.2.6]

Proof. Clearly, every graph constructed as described is 2-connected. Conversely, let a 2-connected graph G be given. Then G contains a

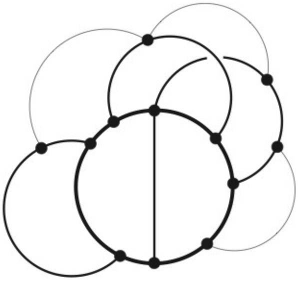

Fig. 3.1.1. The construction of 2-connected graphs

H cycle, and hence has a maximal subgraph H constructible as above. Since any edge $xy \in E(G) \setminus E(H)$ with $x, y \in H$ would define an H-path, H is an induced subgraph of G. Thus if $H \neq G$, then by the connectedness of G there is an edge vw with $v \in G - H$ and $w \in H$. As G is 2-connected, $G - w$ contains a v–H path P. Then wvP is an H-path in G, and $H \cup wvP$ is a constructible subgraph of G larger than H. This contradicts the maximality of H. □

Just as an arbitrary graph can be decomposed into its maximal connected subgraphs, or *components*, we can try to decompose a connected graph G into its maximal 2-connected subgraphs. These may not quite be disjoint, and they may not quite cover all of G. However, it is easy to weaken the notion of 'maximal 2-connected subgraph' slightly so that the subgraphs fitting the weaker notion do cover G and are still nearly disjoint. These 'blocks' fit together nicely in a tree-like fashion, which captures precisely the overall structure of G in terms of those blocks.

block Formally, a *block* is a maximal connected subgraph without a cutvertex.[1] Thus, every block is either a maximal 2-connected subgraph, or a bridge (with its ends), or an isolated vertex. Conversely, every such subgraph is a block. By their maximality, different blocks of G overlap in at most one vertex, which is then a cutvertex of G. Hence every edge of G lies in a unique block, and G is the union of its blocks.

Cycles and bonds are confined to a single block:

[4.6] **Lemma 3.1.2.** *Let G be any graph.*

(i) *The cycles of G are precisely the cycles of its blocks.*

(ii) *The bonds of G are precisely the minimal cuts of its blocks.*

Proof. (i) Any cycle in G is a connected subgraph without a cutvertex, and hence lies in some maximal such subgraph. By definition, this is a block of G.

(ii) The proof follows easily by repeated application of the following observation. Consider any cut in G. Let xy be one of its edges, and B

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[1] . . . of the subgraph; it may contain cutvertices of G.

the block containing it. By the maximality of B in the definition of a block, G contains no B-path. Hence every x–y path of G lies in B, so those edges of our cut that lie in B separate x from y even in G. ☐

As every edge lies in a unique block, belonging to a common block is an equivalence relation on the edge set of a graph. This equivalence can be expressed in two other interesting ways:

Lemma 3.1.3. *The following statements are equivalent for distinct edges e, f of a graph G:* [4.6]

(i) *The edges e, f belong to a common block of G.*

(ii) *The edges e, f belong to a common cycle in G.*

(iii) *The edges e, f belong to a common bond of G.*

Proof. (i)→(ii) It clearly suffices to prove that in a 2-connected graph any two 2-sets of vertices can be joined by two disjoint paths. This follows easily by induction based on Proposition 3.1.1.[2]

(ii)→(iii) Deleting e and f from a cycle $C \ni e, f$ leaves a partition of $V(C)$ into two connected sets. Extend this to a partitition into two connected sets of the vertex set of the component of G containing C. (How?) The edges between these sets form a bond of G containing e and f.

(iii)→(i) By Lemma 3.1.2 (ii), two edges can lie in a common bond only if they belong to the same block. ☐

Our last lemma on blocks shows how they fit together to form the coarse structure of G. Let A denote the set of cutvertices of G, and $\mathcal{B}$ the set of its blocks. We then have a natural bipartite graph on $A \cup \mathcal{B}$ formed by the edges aB with $a \in B$. This *block graph* of G is shown in Figure 3.1.2.

block graph

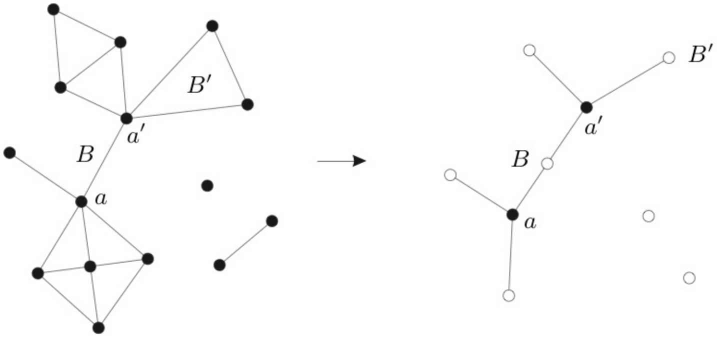

Fig. 3.1.2. A graph and its block graph

Lemma 3.1.4. *The block graph of a connected graph is a tree.* ☐

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[2] See Exercise 5. Note that this is the case $k = 2$ of Menger's theorem (3.3.1).

3.2 The structure of 3-connected graphs

In the last section we saw how every connected graph decomposes canonically into 2-connected subgraphs (and bridges), and how these are arranged in a tree-like way to make up the whole graph. There is a similar canonical decomposition of 2-connected graphs into 3-connected pieces (and cycles), which are again organized in a tree-like way. This non-trivial structure theorem of Tutte is most naturally expressed in terms of *tree-decompositions*, to be introduced in Chapter 12. We therefore omit it here.[3]

Instead, we shall describe how every 3-connected graph can be obtained from a K^4 by a succession of elementary operations preserving 3-connectedness. We then prove a theorem of Tutte about the algebraic structure of the cycle space of 3-connected graphs; this will play an important role again in Chapter 4.5.

Proposition 3.1.1 describes how the 2-connected graphs can be constructed inductively, starting from a cycle. All the graphs constructed in the process were themselves 2-connected, so the graphs constructible in this way are precisely the 2-connected graphs. We shall now do something similar for 3-connected graphs. We shall prove that every 3-connected graph $G \neq K^4$ can be turned into a smaller 3-connected graph in two ways: by deleting an edge (and suppressing any vertices of degree 2 that may arise), and by contracting an edge. Inverting these processes will give us two independent ways of building all 3-connected graphs from a K^4.

$G \dot{-} e$ Given an edge e in a graph G, let us write $G \dot{-} e$ for the multigraph obtained from $G - e$ by suppressing any end of e that has degree 2 in $G - e$. The following lemma can be proved by easy (if tedious) case analysis considering the various positions of potential 2-separators:

Lemma 3.2.1. *Let e be an edge in a graph G. If $G \dot{-} e$ is 3-connected, then so is G.* □

Lemma 3.2.2. *Every 3-connected graph $G \neq K^4$ has an edge e such that $G \dot{-} e$ is another 3-connected graph.*

Proof. We start by showing that G contains a TK^4. Let C be a shortest cycle and $P = u \dots v$ a C-path in G. Then $\mathring{P} \neq \emptyset$ since C is induced, so $G - \{u, v\}$ contains a C–P path Q. Now $C \cup P \cup Q = TK^4$.

J H As $G \neq K^4$, there is a 3-connected graph $J \not\simeq G$ such that G contains a TJ. Choose J with $\|J\|$ maximum, and then $H = TJ \subseteq G$ with $\|H\|$ maximum. We shall find an edge e such that $G \dot{-} e \simeq J$.

Clearly $H \neq G$. Let $P = u \dots v$ be an H-path in G, chosen if possible so that

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[3] The curious reader may take a glance at Exercise 20 of Chapter 12.

u and v do not lie on a common subdivided edge of J. (∗) $P = u \dots v$

If P does *not* satisfy (∗) then $H = J$; for since G is 3-connected, the vertices subdividing an edge of J could be joined by an H-path to a vertex not on the same subdivided edge of J. Moreover, $uv \in E(H)$. Since G has no parallel edges, P has an inner vertex. Now $(H - uv) \cup P$ is another TJ with more edges than H, contradicting our choice of H.

Therefore P does satisfy (∗). Suppressing any vertices of degree 2 in $H \cup P$ we obtain a graph J' such that $J' \dot{-} e = J$, where e is the edge corresponding to P. By (∗) the edge e is not parallel to an edge of J, so J' is indeed a graph. By Lemma 3.2.1, J' is 3-connected. Hence $J' \simeq G$ by the maximality of J, completing the proof. □

Theorem 3.2.3. (Tutte 1966)
A graph G is 3-connected if and only if there exists a sequence $G_0, \dots, G_n$ of graphs such that

(i) *$G_0 = K^4$ and $G_n = G$;*

(ii) *G_{i+1} has an edge e such that $G_i = G_{i+1} \dot{-} e$, for every $i < n$.*

Moreover, the graphs in any such sequence are all 3-connected.

Proof. If G is 3-connected, use Lemma 3.2.2 to find $G_n, \dots, G_0$ in turn. Conversely, if $G_0, \dots, G_n$ is any sequence of graphs satisfying (i) and (ii), then all these graphs, and in particular $G = G_n$, are 3-connected by Lemma 3.2.1. □

Theorem 3.2.3 enables us to construct, recursively, the entire class of 3-connected graphs. Starting from K^4, we simply add to every graph already constructed a new edge in every way compatible with (ii): between two already existing vertices, between newly inserted subdividing vertices (not on the same edge), or between one old vertex and one new subdividing vertex.

We now turn to our second method of reducing 3-connected graphs to K^4, by contracting edges.

Lemma 3.2.4. *Every 3-connected graph $G \neq K^4$ has an edge e such that G/e is again 3-connected.* [4.4.3]

Proof. Suppose there is no such edge e. Then, for every edge $xy \in G$, the graph G/xy contains a separator S of at most 2 vertices. Since $\kappa(G) \geqslant 3$, the contracted vertex v_{xy} of G/xy (see Chapter 1.7) lies in S and $|S| = 2$, i.e. G has a vertex $z \notin \{x, y\}$ such that $\{v_{xy}, z\}$ separates G/xy. Then any two vertices separated by $\{v_{xy}, z\}$ in G/xy are separated in G by $T := \{x, y, z\}$. Since no proper subset of T separates G, every vertex in T has a neighbour in every component C of $G - T$.

xy

z

C

v

w

We choose the edge xy, the vertex z, and the component C so that $|C|$ is as small as possible, and pick a neighbour v of z in C (Fig. 3.2.1). By assumption, G/zv is again not 3-connected, so again there is a vertex w such that $\{z, v, w\}$ separates G, and as before every vertex in $\{z, v, w\}$ has a neighbour in every component of $G - \{z, v, w\}$.

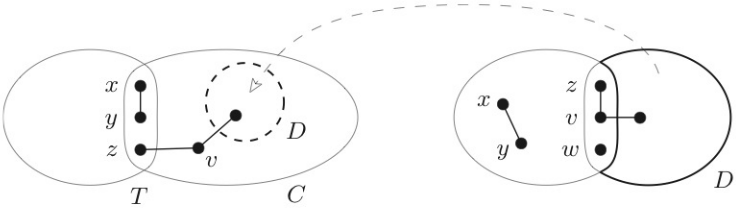

Fig. 3.2.1. Separating vertices in the proof of Lemma 3.2.1

As x and y are adjacent, $G - \{z, v, w\}$ has a component D such that $D \cap \{x, y\} = \emptyset$. Then every neighbour of v in D lies in C (since $v \in C$), so $D \cap C \neq \emptyset$ and hence $D \subsetneqq C$ by the choice of D. This contradicts the choice of xy, z and C. □

Theorem 3.2.5. (Tutte 1961)
A graph G is 3-connected if and only if there exists a sequence $G_0, \ldots, G_n$ of graphs with the following two properties:

(i) $G_0 = K^4$ *and* $G_n = G$;

(ii) G_{i+1} *has an edge* xy *such that* $d(x), d(y) \geqslant 3$ *and* $G_i = G_{i+1}/xy$, *for every* $i < n$.

Moreover, the graphs in any such sequence are all 3-connected.

Proof. If G is 3-connected, then by Lemma 3.2.4 there is a sequence $G_n, \ldots, G_0$ of 3-connected graphs satisfying (i) and (ii).

Conversely, and to show the final statement of the theorem, let $G_0, \ldots, G_n$ be a sequence of graphs satisfying (i) and (ii); we show that if G_i is 3-connected then so is G_{i+1}, for every $i < n$. Suppose not, let S be a separator of at most 2 vertices in G_{i+1}, and let C_1, C_2 be two components of $G_{i+1} - S$. As x and y are adjacent, we may assume that $\{x, y\} \cap V(C_1) = \emptyset$ (Fig. 3.2.2). Then C_2 contains neither both vertices

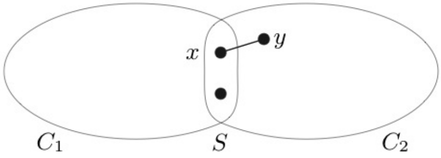

Fig. 3.2.2. The position of $xy \in G_{i+1}$ in the proof of Theorem 3.2.5

x, y nor a vertex $v \notin \{x, y\}$: otherwise v_{xy} or v would be separated from C_1 in G_i by at most two vertices, a contradiction. But now C_2 contains only one vertex: either x or y. This contradicts our assumption of $d(x), d(y) \geqslant 3$. □

Like Theorem 3.2.3, Theorem 3.2.5 enables us to construct all 3-connected graphs inductively from K^4, by simple local alterations and without ever leaving the class of 3-connected graphs. Given a 3-connected graph already constructed, pick any vertex v and split it into two adjacent vertices v', v''; then join these to all the former neighbours of v, each to at least two. This is the essential core of a result of Tutte known as his *wheel theorem*;[4] see the notes for details.

Theorem 3.2.6. (Tutte 1963) [4.5.2]
The cycle space of a 3-connected graph is generated by its non-separating induced cycles.

k

Proof. Let G be a fixed 3-connected graph, of order n say. We prove that each of its cycles C is a sum of non-separating induced cycles, applying induction on $k(C) := n - b$, where b denotes the largest order of a component of $G - C$ if there is one, and $b = 0$ if $V(C) = V(G)$.

C

There are no cycles C for which $k(C) = 0$, so the induction starts. Now let C be given for the induction step. If C is a spanning cycle, it is the sum of two cycles $C_1, C_2 \subseteq C + e$, where e a chord. As $k(C_1), k(C_2) < n = k(C)$, we are home by induction.

B

Assume now that $G - C \neq \emptyset$, and let B be a largest component of $G - C$. Suppose first that

> *$G - B$ contains a C-path $P = u \ldots v$ such that each of the two u–v paths on C has an inner vertex in $N(B)$.* (∗)

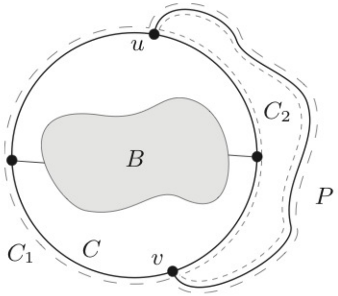

Fig. 3.2.3. C_1 and C_2 are drawn in broken lines

Then C is the sum of the two cycles $C_1, C_2 \subseteq C \cup P$ containing P, and for each of these cycles C_i there is a component of $G - C_i$ that contains

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[4] Graphs of the form $C^n * K^1$ are called *wheels*; thus, K^4 is the smallest wheel. *wheel*

B properly (Fig. 3.2.3). Hence $k(C_i) < k(C)$, and we are again home by induction.

Suppose finally that $(*)$ fails. Then every vertex of C sends an edge to B. (Indeed, if not then C contains an $N(B)$-path $Q = x \dots y$ with $\mathring{Q} \neq \emptyset$. As G is 3-connected, $C - Q \neq \emptyset$, and there is a $\mathring{Q}$–$(C - Q)$ path in $G - \{x, y\}$. Such a path P would satisfy $(*)$.) Since $V(C) = N(B)$, any chord of C would also be a path P as in $(*)$, so C has no chord. Hence unless C itself is induced and non-separating, $G - C$ has a component $B' \neq B$. Let $P = u \dots v$ be a C-path through B', and let Q be a C–P path in $G - \{u, v\}$. Note that Q too avoids B. Now $C \cup P \cup Q$ contains three cycles C_1, C_2, C_3 summing to C and each missing a vertex of C (Fig. 3.2.4). As every vertex of C sends an edge to B, we therefore have $k(C_i) < k(C)$ for every i, completing the induction step. □

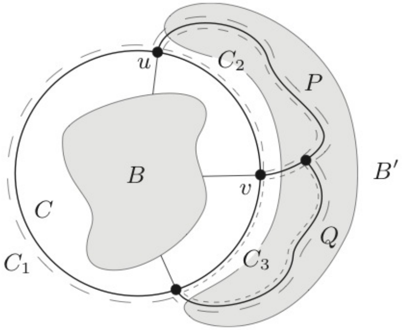

Fig. 3.2.4. Three cycles C_1, C_2, C_3 summing to C, and each missing a vertex of C that sends an edge to B

3.3 Menger's theorem

The following theorem is one of the cornerstones of graph theory.

[3.5.2]
[8.2.5]
[8.4.1]
[12.3.9]
[12.4.4]
[12.4.5]

Theorem 3.3.1. (Menger 1927)
Let $G = (V, E)$ be a graph and $A, B \subseteq V$. Then the minimum number of vertices separating A from B in G is equal to the maximum number of disjoint A–B paths in G.

We offer three proofs. Whenever G, A, B are given as in the theorem, we denote by $k = k\,(G, A, B)$ the minimum number of vertices separating A from B in G. Clearly, G cannot contain more than k disjoint A–B paths; our task will be to show that k such paths exist. k

First proof. We apply induction on $\|G\|$. If G has no edge, then $|A \cap B| = k$ and we have k trivial A–B paths. So we assume that G has an edge $e = xy$. If G has no k disjoint A–B paths, then neither does G/e; here, we count the contracted vertex v_e as an element of A (resp. B) in G/e if in G at least one of x, y lies in A (resp. B). By the induction hypothesis, G/e contains an A–B separator Y of fewer than k vertices. Among these must be the vertex v_e, since otherwise $Y \subseteq V$ would be an A–B separator in G. Then $X := (Y \setminus \{v_e\}) \cup \{x, y\}$ is an A–B separator in G of exactly k vertices.

We now consider the graph $G - e$. Since $x, y \in X$, every A–X separator in $G - e$ is also an A–B separator in G and hence contains at least k vertices. So by induction there are k disjoint A–X paths in $G - e$, and similarly there are k disjoint X–B paths in $G - e$. As X separates A from B, these two path systems do not meet outside X, and can thus be combined to k disjoint A–B paths. □

exceeds

Let $\mathcal{P}$ be a set of disjoint A–B paths, and let $\mathcal{Q}$ be another such set. We say that $\mathcal{Q}$ *exceeds* $\mathcal{P}$ if the set of vertices in A that lie on a path in $\mathcal{P}$ is a proper subset of the set of vertices in A that lie on a path in $\mathcal{Q}$, and likewise for B. Then, in particular, $|\mathcal{Q}| \geqslant |\mathcal{P}| + 1$.

Second proof. We prove the following stronger statement:

> *If $\mathcal{P}$ is any set of fewer than k disjoint A–B paths in G, then G contains a set of $|\mathcal{P}| + 1$ disjoint A–B paths exceeding $\mathcal{P}$.*

Keeping G and A fixed, we let B vary and apply induction on $|\bigcup \mathcal{P}|$. Let R be an A–B path that avoids the (fewer than k) vertices of B that lie on a path in $\mathcal{P}$. If R avoids all the paths in $\mathcal{P}$, then $\mathcal{P} \cup \{R\}$ exceeds $\mathcal{P}$, as desired. (This will happen when $\mathcal{P} = \emptyset$, so the induction starts.) If not, let x be the last vertex of R that lies on some $P \in \mathcal{P}$ (Fig. 3.3.1).

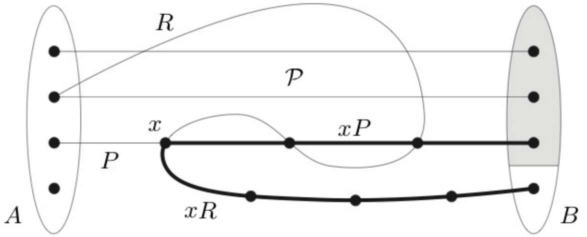

Fig. 3.3.1. Paths in the second proof of Menger's theorem

Put

$$B' := B \cup V(xP \cup xR) \quad \text{and} \quad \mathcal{P}' := \big(\mathcal{P} \setminus \{P\}\big) \cup \{Px\}.$$

Then $|\mathcal{P}'| = |\mathcal{P}|$ (but $|\bigcup \mathcal{P}'| < |\bigcup \mathcal{P}|$) and $k(G, A, B') \geqslant k(G, A, B)$, so by the induction hypothesis there is a set $\mathcal{Q}'$ of $|\mathcal{P}'| + 1$ disjoint A–B' paths exceeding $\mathcal{P}'$. Then $\mathcal{Q}'$ contains a path Q ending in x, and a unique path Q' whose last vertex y is not among the last vertices of the paths in $\mathcal{P}'$.

If $y \notin xP$, we let $\mathcal{Q}$ be obtained from $\mathcal{Q}'$ by adding xP to Q, and adding yR to Q' if $y \notin B$. Otherwise $y \in \mathring{x}P$, and we let $\mathcal{Q}$ be obtained from $\mathcal{Q}'$ by adding xR to Q and adding yP to Q'. In all cases $\mathcal{Q}$ exceeds $\mathcal{P}$, as desired. □

Applied to a bipartite graph, Menger's theorem specializes to the assertion of König's theorem (2.1.1). For our third proof, we shall adapt the alternating path proof of König's theorem to the more general set-up of Theorem 3.3.1. Let again G, A, B be given, and let $\mathcal{P}$ be a set of disjoint A–B paths in G. Let us say that an A–B separator $X \subseteq V$ lies *on* $\mathcal{P}$ if it consists of a choice of exactly one vertex from each path in $\mathcal{P}$. If we can find such a separator X, then clearly $k \leqslant |X| = |\mathcal{P}|$, and Menger's theorem will be proved.

$\mathcal{P}$

on

Put

$$V[\mathcal{P}] := \bigcup \{\, V(P) \mid P \in \mathcal{P}\,\}$$

$$E[\mathcal{P}] := \bigcup \{\, E(P) \mid P \in \mathcal{P}\,\}.$$

W, x_i, e_i

alternating walk

Let a walk $W = x_0 e_0 x_1 e_1 \dots e_{n-1} x_n$ in G with $e_i \neq e_j$ for $i \neq j$ be said to *alternate* with respect to $\mathcal{P}$ (Fig. 3.3.2) if it starts in $A \setminus V[\mathcal{P}]$ and the following three conditions hold for all $i < n$ (with $e_{-1} := e_0$ in (iii)):

(i) if $e_i = e \in E[\mathcal{P}]$, then W traverses the edge e backwards, i.e. $x_{i+1} \in P\mathring{x}_i$ for some $P \in \mathcal{P}$;

(ii) if $x_i = x_j$ with $i \neq j$, then $x_i \in V[\mathcal{P}]$;

(iii) if $x_i \in V[\mathcal{P}]$, then $\{e_{i-1}, e_i\} \cap E[\mathcal{P}] \neq \emptyset$.

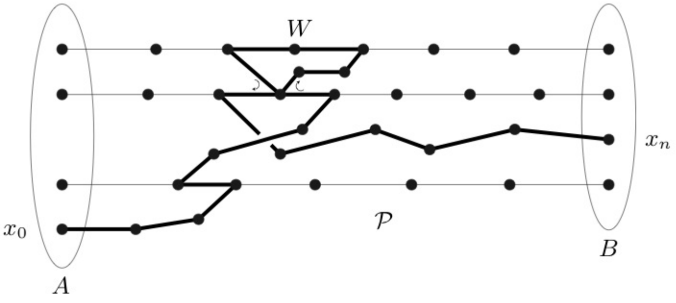

Fig. 3.3.2. An alternating walk from A to B

Note that, by (ii), any vertex outside $V[\mathcal{P}]$ occurs at most once on W. And since the edges e_i of W are all distinct, (iii) implies that any vertex $v \in V[\mathcal{P}]$ occurs at most twice on W. For $v \neq x_n$, this can happen in exactly the following two ways. If $x_i = x_j$ with $0 < i < j < n$, then

$$\begin{aligned} \textit{either}\ & e_{i-1}, e_j \in E[\mathcal{P}] \text{ and } e_i, e_{j-1} \notin E[\mathcal{P}] \\ \textit{or}\ & e_i, e_{j-1} \in E[\mathcal{P}] \text{ and } e_{i-1}, e_j \notin E[\mathcal{P}]\,. \end{aligned}$$

Unless otherwise stated, any use of the word 'alternate' below will refer to our fixed path system $\mathcal{P}$.

The next two lemmas together make up our third proof of Menger's theorem. We state and prove them in a way that makes them reusable in Chapter 8, when we prove Menger's theorem for infinite graphs.

Lemma 3.3.2. *If an alternating walk W as above ends in $B \setminus V[\mathcal{P}]$, then G contains a set of disjoint A–B paths exceeding $\mathcal{P}$.* [8.4.5]

Proof. We may assume that W has only its first vertex in $A \setminus V[\mathcal{P}]$ and only its last vertex in $B \setminus V[\mathcal{P}]$. Let H be the graph on $V(G)$ whose edge set is the symmetric difference of $E[\mathcal{P}]$ with $\{e_0, \dots, e_{n-1}\}$. In H, the ends of the paths in $\mathcal{P}$ and of W have degree 1 (or 0, if the path or W is trivial), and all other vertices have degree 0 or 2.

For each vertex $a \in (A \cap V[\mathcal{P}]) \cup \{x_0\}$, therefore, the component of H containing a is a path, $P = v_0 \dots v_k$ say, which starts in a and ends in A or B. Using conditions (i) and (iii), one easily shows by induction on $i = 0, \dots, k-1$ that P traverses each of its edges $e = v_i v_{i+1}$ in the forward direction with respect to $\mathcal{P}$ or W. (Formally: if $e \in P'$ with $P' \in \mathcal{P}$, then $v_i \in P' \mathring{v}_{i+1}$; if $e = e_j \in W$, then $v_i = x_j$ and $v_{i+1} = x_{j+1}$.) Hence, P is an A–B path. (When G is infinite, this last conclusion uses the fact that W meets only finitely many paths in $\mathcal{P}$, and hence every component of H is finite.)

Similarly, for every $b \in (B \cap V[\mathcal{P}]) \cup \{x_n\}$ there is an A–B path in H that ends in b. The set of A–B paths in H therefore exceeds $\mathcal{P}$. □

Lemma 3.3.3. *If no alternating walk W as above ends in $B \setminus V[\mathcal{P}]$, then G contains an A–B separator on $\mathcal{P}$.* [8.4.5]

Proof. Let

$$A_1 := A \cap V[\mathcal{P}] \quad \text{and} \quad A_2 := A \setminus A_1\,,$$ A_1, A_2

and

$$B_1 := B \cap V[\mathcal{P}] \quad \text{and} \quad B_2 := B \setminus B_1\,.$$ B_1, B_2

For every path $P \in \mathcal{P}$, let x_P be the last vertex of P that lies on some alternating walk; if no such vertex exists, let x_P be the first vertex of P. Our aim is to show that x_P

$$X := \{\, x_P \mid P \in \mathcal{P} \,\}$$ X

meets every A–B path in G; then X is an A–B separator on $\mathcal{P}$.

Q

Suppose there is an A–B path Q that avoids X. We know that Q meets $V[\mathcal{P}]$, as otherwise it would be an alternating walk ending in B_2. Now the A–$V[\mathcal{P}]$ path in Q is either an alternating walk or consists only of the first vertex of some path in $\mathcal{P}$. Therefore Q also meets the vertex set $V[\mathcal{P}']$ of

$\mathcal{P}'$

$$\mathcal{P}' := \{\, Px_P \mid P \in \mathcal{P} \,\}\,.$$

y, P, x

W

Let y be the last vertex of Q in $V[\mathcal{P}']$, say $y \in P \in \mathcal{P}$, and let $x := x_P$. As Q avoids X and hence x, we have $y \in P\mathring{x}$. In particular, $x = x_P$ is not the first vertex of P, and so there is an alternating walk W ending at x. Then $W \cup xPyQ$ is a walk from A_2 to B (Fig. 3.3.3). If this walk alternates and ends in B_2, we have our desired contradiction.

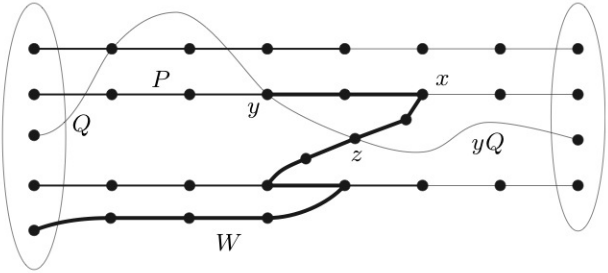

Fig. 3.3.3. Alternating walks in the proof of Lemma 3.3.3.

x', W'

How could $W \cup xPyQ$ fail to alternate? For example, W might already use an edge of xPy. But if x' is the first vertex of W on $xP\mathring{y}$, then $W' := Wx'Py$ is an alternating walk from A_2 to y. (By Wx' we mean the initial segment of W ending at the first occurrence of x' on W; from there, W' follows P back to y.) Even our new walk $W'yQ$ need not yet alternate: for example, W' might still meet $\mathring{y}Q$. By definition of $\mathcal{P}'$ and W, however, and the choice of y on Q, we have

$$V(W') \cap V[\mathcal{P}] \subseteq V[\mathcal{P}'] \quad \text{and} \quad V(\mathring{y}Q) \cap V[\mathcal{P}'] = \emptyset\,.$$

Thus, W' and $\mathring{y}Q$ can meet only outside $\mathcal{P}$.

z

W''

If W' does indeed meet $\mathring{y}Q$, we let z be the first vertex of W' on $\mathring{y}Q$ and set $W'' := W'zQ$. Otherwise we set $W'' := W' \cup yQ$. In both cases W'' alternates with respect to $\mathcal{P}'$, because W' does and $\mathring{y}Q$ avoids $V[\mathcal{P}']$. (W'' satisfies condition (iii) at y in the second case even if y occurs twice on W', because W'' then contains the entire walk W' and not just its initial segment $W'y$.) By definition of $\mathcal{P}'$, therefore, W'' avoids $V[\mathcal{P}] \setminus V[\mathcal{P}']$. Thus W'' also alternates with respect to $\mathcal{P}$ and ends in B_2, contrary to our assumptions. □

Third proof of Menger's theorem. Let $\mathcal{P}$ contain as many disjoint A–B paths in G as possible. Then by Lemma 3.3.2, no alternating walk ends in $B \smallsetminus V[\mathcal{P}]$. By Lemma 3.3.3, this implies that G has an A–B separator X on $\mathcal{P}$, giving $k \leqslant |X| = |\mathcal{P}|$ as desired. $\square$

A set of a–B paths is called an a–B *fan* if any two of the paths have only a in common. *fan*

Corollary 3.3.4. *For $B \subseteq V$ and $a \in V \smallsetminus B$, the minimum number of vertices $\neq a$ separating a from B in G is equal to the maximum number of paths forming an a–B fan in G.* [10.1.2]

Proof. Apply Theorem 3.3.1 to $G - a$ with $A := N_G(a)$. $\square$

Corollary 3.3.5. *Let a and b be two distinct vertices of G.*

(i) *If $ab \notin E$, then the minimum number of vertices $\neq a, b$ separating a from b in G is equal to the maximum number of independent a–b paths in G.*

(ii) *The minimum number of edges separating a from b in G is equal to the maximum number of edge-disjoint a–b paths in G.*

Proof. (i) Apply Theorem 3.3.1 to $G - \{a, b\}$, with $A := N_G(a)$ and $B := N_G(b)$.

(ii) Apply Theorem 3.3.1 to the line graph of G, with $A := E(a)$ and $B := E(b)$. $\square$

[4.2.7]
[6.6.1]
[9.4.2]

Theorem 3.3.6. (Global Version of Menger's Theorem)

(i) *A graph is k-connected if and only if it contains k independent paths between any two vertices.*

(ii) *A graph is k-edge-connected if and only if it contains k edge-disjoint paths between any two vertices.*

Proof. (i) If a graph G contains k independent paths between any two vertices, then $|G| > k$ and G cannot be separated by fewer than k vertices; thus, G is k-connected.

Conversely, suppose that G is k-connected (and, in particular, has more than k vertices) but contains vertices a, b not linked by k independent paths. By Corollary 3.3.5 (i), a and b are adjacent; let $G' := G - ab$. Then G' contains at most $k - 2$ independent a–b paths. By Corollary 3.3.5 (i), we can separate a and b in G' by a set X of at most $k - 2$ vertices. As $|G| > k$, there is at least one further vertex $v \notin X \cup \{a, b\}$ in G. Now X separates v in G' from either a or b—say, from a. But then $X \cup \{b\}$ is a set of at most $k - 1$ vertices separating v from a in G, contradicting the k-connectedness of G. a, b G' X v

(ii) follows straight from Corollary 3.3.5 (ii). $\square$

3.4 Mader's theorem

In analogy to Menger's theorem we may consider the following question: given a graph G with an induced subgraph H, up to how many independent H-paths can we find in G?

In this section, we present without proof a deep theorem of Mader, which solves the above problem in a fashion similar to Menger's theorem. Again, the theorem says that an upper bound on the number of such paths that arises naturally from the size of certain separators is indeed attained by some suitable set of paths.

X
F

What could such an upper bound look like? Clearly, if $X \subseteq V(G-H)$ and $F \subseteq E(G-H)$ are such that every H-path in G has a vertex or an edge in $X \cup F$, then G cannot contain more than $|X \cup F|$ independent H-paths. Hence, the least cardinality of such a set $X \cup F$ is a natural upper bound for the maximum number of independent H-paths. (Note that every H-path meets $G-H$, because H is induced in G and edges of H do not count as H-paths.)

$\mathcal{C}_F$
∂C

In contrast to Menger's theorem, this bound can still be improved. The minimality of $|X \cup F|$ implies that no edge in F has an end in X: otherwise this edge would not be needed in the separator. Let $Y := V(G-H) \smallsetminus X$, and denote by $\mathcal{C}_F$ the set of components of the graph (Y, F). Since every H-path avoiding X contains an edge from F, it has at least two vertices in ∂C for some $C \in \mathcal{C}_F$, where ∂C denotes the set of vertices in C with a neighbour in $G-X-C$ (Fig. 3.4.1). The number of independent H-paths in G is therefore bounded above by

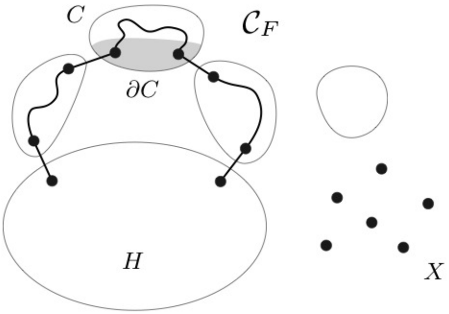

Fig. 3.4.1. An H-path in $G-X$

$M_G(H)$

$$M_G(H) := \min \left(|X| + \sum_{C \in \mathcal{C}_F} \left\lfloor \tfrac{1}{2} \, |\partial C| \right\rfloor \right),$$

X

where the minimum is taken over all X and F as described above: $X \subseteq V(G-H)$ and $F \subseteq E(G-H-X)$ such that every H-path in G has a vertex or an edge in $X \cup F$.

Now Mader's theorem says that this upper bound is always attained by some set of independent H-paths:

Theorem 3.4.1. (Mader 1978)
Given a graph G with an induced subgraph H, there are always $M_G(H)$ independent H-paths in G.

In order to obtain direct analogues to the vertex and edge version of Menger's theorem, let us consider the two special cases of the above problem where either F or X is required to be empty. Given an induced subgraph $H \subseteq G$, we denote by $\kappa_G(H)$ the least cardinality of a vertex set $X \subseteq V(G-H)$ that meets every H-path in G. Similarly, we let $\lambda_G(H)$ denote the least cardinality of an edge set $F \subseteq E(G)$ that meets every H-path in G. $\kappa_G(H)$ $\lambda_G(H)$

Corollary 3.4.2. *Given a graph G with an induced subgraph H, there are at least $\frac{1}{2}\kappa_G(H)$ independent H-paths and at least $\frac{1}{2}\lambda_G(H)$ edge-disjoint H-paths in G.*

Proof. To prove the first assertion, let k be the maximum number of independent H-paths in G. By Theorem 3.4.1, there are sets $X \subseteq V(G-H)$ and $F \subseteq E(G-H-X)$ with k

$$k = |X| + \sum_{C \in \mathcal{C}_F} \left\lfloor \tfrac{1}{2}\,|\partial C| \right\rfloor$$

such that every H-path in G has a vertex in X or an edge in F. For every $C \in \mathcal{C}_F$ with $\partial C \neq \emptyset$, pick a vertex $v \in \partial C$ and let $Y_C := \partial C \setminus \{v\}$; if $\partial C = \emptyset$, let $Y_C := \emptyset$. Then $\left\lfloor \frac{1}{2}\,|\partial C| \right\rfloor \geqslant \frac{1}{2}\,|Y_C|$ for all $C \in \mathcal{C}_F$. Moreover, for $Y := \bigcup_{C \in \mathcal{C}_F} Y_C$ every H-path has a vertex in $X \cup Y$. Hence Y

$$k \geqslant |X| + \sum_{C \in \mathcal{C}_F} \tfrac{1}{2}\,|Y_C| \geqslant \tfrac{1}{2}\,|X \cup Y| \geqslant \tfrac{1}{2}\kappa_G(H)$$

as claimed.

The second assertion follows from the first by considering the line graph of G (Exercise 21). □

It may come as a surprise to see that the bounds in Corollary 3.4.2 are best possible (as general bounds): one can find examples for G and H where G contains no more than $\frac{1}{2}\kappa_G(H)$ independent H-paths or no more than $\frac{1}{2}\lambda_G(H)$ edge-disjoint H-paths (Exercises 22 and 23).

3.5 Linking pairs of vertices

Let G be a graph, and let $X \subseteq V(G)$ be a set of vertices. We call X *linked* in G if whenever we pick distinct vertices $s_1, \dots, s_\ell$, $t_1, \dots, t_\ell$ in X we can find disjoint paths $P_1, \dots, P_\ell$ in G such that each P_i links s_i to t_i and has no inner vertex in X. Thus, unlike in Menger's theorem, we are not merely asking for disjoint paths between two *sets* of vertices: we insist that each of these paths shall link a specified pair of endvertices.

linked

If $|G| \geqslant 2k$ and every set X of at most $2k$ vertices is linked in G, then G is k*-linked*. Clearly, this is equivalent to requiring merely that $|G| \geqslant 2k$ and disjoint paths $P_i = s_i \dots t_i$ exist for every choice of *exactly* $2k$ distinct vertices $s_1, \dots, s_k$, $t_1, \dots, t_k$: just add dummy vertices to X to bring it up to size $2k$. In practice, the latter is easier to prove, because we need not worry about inner vertices in X.

k-linked

Clearly, every k-linked graph is k-connected. The converse, however, seems far from true: being k-linked is clearly a much stronger property than k-connectedness. Still, we shall prove in this section that we can force a graph to be k-linked by assuming that it is $f(k)$-connected, for some function $f\colon \mathbb{N} \to \mathbb{N}$. We first give a nice and simple proof that such a function f exists at all. In the remainder of the section we then prove that f can even be chosen linear.

The basic idea in the simple proof is as follows. If we can prove that G contains a subdivision K of a large complete graph, we can use Menger's theorem to link the vertices of X disjointly to branch vertices of K, and then hope to pair them up as desired through the subdivided edges of K. This requires, of course, that our paths do not hit too many of the subdivided edges before reaching the branch vertices of K.

To show that K exists is a lemma which more properly belongs in Chapter 7, and we shall derive an improved version there from the linearity theorem (3.5.3) proved later in this section. Instead of assuming high connectivity, it suffices that G has large enough average degree (cf. Theorem 1.4.3):

Lemma 3.5.1. *There is a function $h\colon \mathbb{N} \to \mathbb{N}$ such that every graph of average degree at least $h(r)$ contains K^r as a topological minor, for every $r \in \mathbb{N}$.*

(1.2.2)
(1.3.1)

Proof. For $r \leqslant 2$, the assertion holds with $h(r) = 1$; we now assume that $r \geqslant 3$. We show by induction on $m = r, \dots, \binom{r}{2}$ that every graph G with average degree $d(G) \geqslant 2^m$ has a topological minor X with r vertices and m edges; for $m = \binom{r}{2}$ this implies the assertion with $h(r) = 2^{\binom{r}{2}}$.

If $m = r$ then, by Propositions 1.2.2 and 1.3.1, G contains a cycle of length at least $\varepsilon(G) + 1 \geqslant 2^{r-1} + 1 \geqslant r + 1$, and the assertion follows with $X = C^r$.

Now let $r < m \leqslant \binom{r}{2}$, and assume the assertion holds for smaller m. Let G with $d(G) \geqslant 2^m$ be given; thus, $\varepsilon(G) \geqslant 2^{m-1}$. Since G has a

component C with $\varepsilon(C) \geqslant \varepsilon(G)$, we may assume that G is connected.
Consider a maximal set $U \subseteq V(G)$ such that U is connected in G and U
$\varepsilon(G/U) \geqslant 2^{m-1}$; such a set U exists, because G itself has the form G/U
with $|U| = 1$. Since G is connected, we have $N(U) \neq \emptyset$.

Let $H := G[N(U)]$. If H has a vertex v of degree $d_H(v) < 2^{m-1}$, we H
may add it to U and obtain a contradiction to the maximality of U: when
we contract the edge vv_U in G/U, we lose one vertex and $d_H(v)+1 \leqslant$
2^{m-1} edges, so ε will still be at least 2^{m-1}. Therefore $d(H) \geqslant \delta(H) \geqslant$
2^{m-1}. By the induction hypothesis, H contains a TY with $|Y| = r$
and $\|Y\| = m-1$. Let x, y be two branch vertices of this TY that are
non-adjacent in Y. Since x and y lie in $N(U)$ and U is connected in G,
G contains an x–y path whose inner vertices lie in U. Adding this path
to the TY, we obtain the desired TX. □

Theorem 3.5.2. (Jung 1970; Larman & Mani 1970)
There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every $f(k)$-connected graph is k-linked, for all $k \in \mathbb{N}$.

Proof. We prove the assertion for $f(k) = h(3k) + 2k$, where h is a (3.3.1)
function as in Lemma 3.5.1. Let G be an $f(k)$-connected graph. Then G
$d(G) \geqslant \delta(G) \geqslant \kappa(G) \geqslant h(3k)$; let K be a TK^{3k} in G as provided by K
Lemma 3.5.1, and let U denote its set of branch vertices. U

For the proof that G is k-linked, let distinct vertices $s_1, \dots, s_k$ s_i, t_i
and $t_1, \dots, t_k$ be given. By definition of $f(k)$, we have $\kappa(G) \geqslant 2k$.
Hence by Menger's theorem (3.3.1), G contains disjoint paths $P_1, \dots, P_k$,
$Q_1, \dots, Q_k$, such that each P_i starts in s_i, each Q_i starts in t_i, and all P_i, Q_i
these paths end in U but have no inner vertices in U. Let the set $\mathcal{P}$ of $\mathcal{P}$
these paths be chosen so that their total number of edges outside $E(K)$
is as small as possible.

Let $u_1, \dots, u_k$ be those k vertices in U that are not an end of a path in $\mathcal{P}$. For each $i = 1, \dots, k$, let L_i be the U-path in K (i.e., the subdivided edge of the K^{3k}) from u_i to the end of P_i in U, and let v_i be the first vertex of L_i on any path $P \in \mathcal{P}$. By definition of $\mathcal{P}$, P has no more edges outside $E(K)$ than $Pv_iL_iu_i$ does, so $v_iP = v_iL_i$ and hence $P = P_i$ (Fig. 3.5.1). Similarly, if M_i denotes the U-path in K from u_i

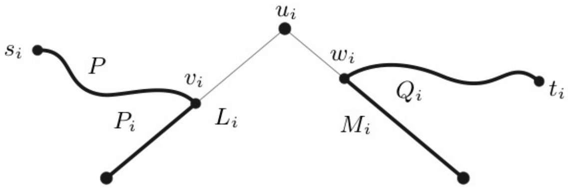

Fig. 3.5.1. Constructing an s_i–t_i path via u_i

to the end of Q_i in U, and w_i denotes the first vertex of M_i on any path in $\mathcal{P}$, then this path is Q_i. Then the paths $s_iP_iv_iL_iu_iM_iw_iQ_it_i$ are disjoint for different i and show that G is k-linked. □

The proof of Theorem 3.5.2 yields only an exponential upper bound for the function $f(k)$. As $2\varepsilon(G) \geqslant \delta(G) \geqslant \kappa(G)$, the following result implies the linear bound of $f(k) = 16k$:

[7.2.1] **Theorem 3.5.3.** (Thomas & Wollan 2005)
Let G be a graph and $k \in \mathbb{N}$. If G is $2k$-connected and $\varepsilon(G) \geqslant 8k$, then G is k-linked.

We begin our proof of Theorem 3.5.3 with a lemma.

Lemma 3.5.4. *If $\delta(G) \geqslant 8k$ and $|G| \leqslant 16k$, then G has a k-linked subgraph.*

Proof. If G itself is k-linked there is nothing to show, so suppose not. X Then we can find a set X of $2k$ vertices $s_1, \dots, s_k, t_1, \dots, t_k$ that cannot $\mathcal{P}$ be linked in G by disjoint paths $P_i = s_i \dots t_i$. Let $\mathcal{P}$ be a set of as many such paths as possible, without inner vertices in X and all of length at most 7. If there are several such sets $\mathcal{P}$, we choose one with $|\bigcup \mathcal{P}|$ $s_1,\ t_1$ minimum. We may assume that $\mathcal{P}$ contains no path from s_1 to t_1. Let J be the subgraph of G induced by X and all the vertices on the paths J, H in $\mathcal{P}$, and let $H := G - J$.

Note that each vertex $v \in H$ has at most three neighbours on any given $P_i \in \mathcal{P}$: if it had four, then replacing the segment uP_iw between its first and its last neighbour on P_i by the path uvw would reduce $|\bigcup \mathcal{P}|$ and thus contradict our choice of $\mathcal{P}$. So v has at most 3 neighbours in J for every $i = 1, \dots, k$, at most $3k$ in total. As $\delta(G) \geqslant 8k$ by assumption, as well as $|G| \leqslant 16k$ and $|X| = 2k$, we deduce that

$$\delta(H) \geqslant 5k \quad \text{and} \quad |H| \leqslant 14k\,. \tag{1}$$

Our next aim is to show that H is disconnected. Since each of the paths in $\mathcal{P}$ has at most eight vertices, we have $|J - \{s_1, t_1\}| \leqslant 8(k-1)$. Therefore s_1 has a neighbour s in H, and t_1 has a neighbour t in H. Put $S := \{\, s' \in H \mid d_H(s, s') \leqslant 2 \,\}$ and $T := \{\, t' \in H \mid d_H(t, t') \leqslant 2 \,\}$. Since $G - \bigcup \mathcal{P}$ contains no s_1–t_1 path of length at most 7, we have $S \cap T = \emptyset$, and there is no S–T edge in H. To prove that H is disconnected, it thus suffices to show that $V(H) = S \cup T$. But for any vertex $v \in H - (S \cup T)$ the sets $N_H(s)$, $N_H(t)$ and $N_H(v)$ are disjoint and each have size at least $5k$, contradicting (1).

So H is disconnected; let C be its smaller component. By (1),

$$2\delta(C) \geqslant 2\delta(H) \geqslant 7k + 3k \geqslant \tfrac{1}{2}\,|H| + 3k \geqslant |C| + 3k\,. \tag{2}$$

We complete the proof by showing that C is k-linked. As $\delta(C) \geqslant 5k$, we have $|C| \geqslant 2k$. Let Y be a set of at most $2k$ vertices in C. By (2), every two vertices in Y have at least $3k$ common neighbours, at least k of which lie outside Y. We can therefore link any desired $\ell \leqslant k$ pairs of vertices in Y inductively by paths of length 2 whose inner vertex lies outside Y. □

Before we launch into the proof of Theorem 3.5.3, let us look at its main ideas. To prove that G is k-linked, we have to consider a given set X of up to $2k$ vertices and show that X is linked in G. Ideally, we would like to use Lemma 3.5.4 to find a linked subgraph L somewhere in G, and then use our assumption of $\kappa(G) \geqslant 2k$ to obtain a set of $|X|$ disjoint X–L paths by Menger's theorem (3.3.1). Then X could be linked via these paths and L, completing the proof.

Unfortunately, we cannot expect to find a subgraph H such that $\delta(H) \geqslant 8k$ and $|H| \leqslant 16k$ (in which L could be found by Lemma 3.5.4); cf. Corollary 11.2.3. However, it is not too difficult to find a minor $H \preccurlyeq G$ that has such a subgraph (Ex. 20, Ch. 7), even so that the vertices of X come to lie in distinct branch sets of H. We may then regard X as a subset of $V(H)$, and Lemma 3.5.4 provides us with a linked subgraph L of H. The only problem now is that H need no longer be $2k$-connected, that is, our assumption of $\kappa(G) \geqslant 2k$ will not ensure that we can link X to L by $|X|$ disjoint paths in H.

And here comes the clever bit of the proof: it relaxes the assumption of $\kappa \geqslant 2k$ to a weaker assumption that does get passed on to H. This weaker assumption is that *if* we can separate X from another part of G (or H) by fewer than $|X|$ vertices, then this other part must be 'light': roughly, its own value of ε must not exceed $8k$. If X then fails to link to L by $|X|$ disjoint paths, and hence H has a separation $\{A, B\}$ with $X \subseteq A$ and $L \subseteq B$ and $|A \cap B| < |X|$, we know that ε is still at least $8k$ on $H[A]$, because the B-part of H was light.

The idea now is to continue the proof inside $H' := H[A]$ by induction. This still needs some ingenuity, since it is not enough that ε is large on H': we also need that for every low-order separation (A', B') of H' with $X \subseteq A'$ the B'-part is light. That need not be true. But when it fails, we shall be able to use induction on $H'[B']$ to show that $A' \cap B'$ is linked in $H'[B']$, and use this for our proof that X is linked in H.

X-separation

Given $k \in \mathbb{N}$, a graph G, and $A, B, X \subseteq V(G)$, call the ordered pair (A, B) an *X-separation* of G if $\{A, B\}$ is a proper separation of G of order at most $|X|$ and $X \subseteq A$. An X-separation (A, B) is *small* if $|A \cap B| < |X|$, and *linked* if $A \cap B$ is linked in $G[B]$.

small/linked

$\| \ \|^+$

light

heavy

Call a set $U \subseteq V(G)$ *light* in G if $\|U\|^+ \leqslant 8k\,|U|$, where $\|U\|^+$ denotes the number of edges of G with at least one end in U. A set of vertices is *heavy* if it is not light.

k

Proof of Theorem 3.5.3. We shall prove the following, for fixed $k \in \mathbb{N}$:

$G = (V, E)$
X

Let $G = (V, E)$ be a graph and $X \subseteq V$ a set of at most $2k$ vertices. If $V \smallsetminus X$ is heavy and for every small X-separation (A, B) the set $B \smallsetminus A$ is light, then X is linked in G. $(*)$

To see that $(*)$ implies the theorem, assume that $\kappa(G) \geqslant 2k$ and $\varepsilon(G) \geqslant 8k$, and let X be a set of exactly $2k$ vertices. Then G has no small X-separation. And $V \smallsetminus X$ is heavy, since

$$\|V \smallsetminus X\|^+ \geqslant \|G\| - \binom{2k}{2} > 8k\,|V| - 16k^2 = 8k\,|V \smallsetminus X|\,.$$

By $(*)$, X is linked in G, completing the proof that G is k-linked.

We prove $(*)$ by induction on $|G|$, and for each value of $|G|$ by induction on $\|V \smallsetminus X\|^+$. If $|G| = 1$ then X is linked in G. For the induction step, let G and X be given as in $(*)$. We first prove the following:

We may assume that G has no linked X-separation. (1)

(A, B)
S

For our proof of (1), suppose that G has a linked X-separation (A, B). Let us choose one with A minimal, and put $S := A \cap B$.

We first consider the case that $|S| = |X|$. If $G[A]$ contains $|X|$ disjoint X–S paths, then X is linked in G because (A, B) is linked, completing the proof of $(*)$. If not, then by Menger's theorem (3.3.1) $G[A]$ has a small X-separation (A', B') such that $B' \supseteq S$. If we choose this of minimum order, i.e. with $|A' \cap B'|$ minimum, we can link $A' \cap B'$ to S in $G[B']$ by $|A' \cap B'|$ disjoint paths, again by Menger's theorem. But then $(A', B' \cup B)$ is a linked X-separation of G that contradicts the choice of (A, B).

G'

So $|S| < |X|$. Let G' be obtained from $G[A]$ by adding any missing edges on S, so that $G'[S]$ is a complete subgraph of G'. As (A, B) is now a small X-separation, our assumption in $(*)$ says that $B \smallsetminus A$ is light in G. Thus, G' arises from G by deleting $|B \smallsetminus A|$ vertices outside X and at most $8k\,|B \smallsetminus A|$ edges, and possibly adding some edges. As $V \smallsetminus X$ is heavy in G, this implies that

$A \smallsetminus X$ is heavy in G'.

In order to be able to apply the induction hypothesis to G', let us show next that for every small X-separation (A', B') of G' the set $B' \smallsetminus A'$ is light in G'. Suppose not, and choose a counterexample (A', B') with B' minimal. As $G'[S]$ is complete, we have $S \subseteq A'$ or $S \subseteq B'$.

(A', B')

If $S \subseteq A'$ then $B' \cap B \subseteq S \subseteq A'$, so $(A' \cup B, B')$ is a small X-separation of G. Moreover,

$$B' \smallsetminus (A' \cup B) = B' \smallsetminus A',$$

and no edge of $G' - E$ on S is incident with this set (Fig 3.5.2). Our assumption that this set is heavy in G', by the choice of (A', B'), therefore implies that it is heavy also in G. As $(A' \cup B, B')$ is a small X-separation of G, this contradicts our assumptions in $(*)$.

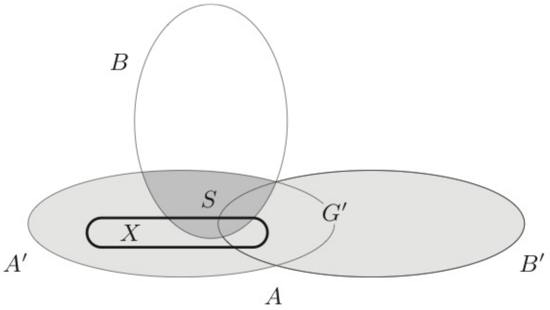

Fig. 3.5.2. If $S \subseteq A'$, then $(A' \cup B, B')$ is an X-separation of G

Hence $S \subseteq B'$. By our choice of (A', B'), the graph $G'' := G'[B']$ satisfies the premise of $(*)$ for $X'' := A' \cap B'$. Indeed, $B' \setminus X'' = B' \setminus A'$ is heavy, and by the minimality of B' any small X''-separation (A'', B'') of G'' will be such that $B'' \setminus A''$ is light, because $(A' \cup A'', B'')$ will be a small X-separation of G', and $B'' \setminus A'' = B'' \setminus (A' \cup A'')$.

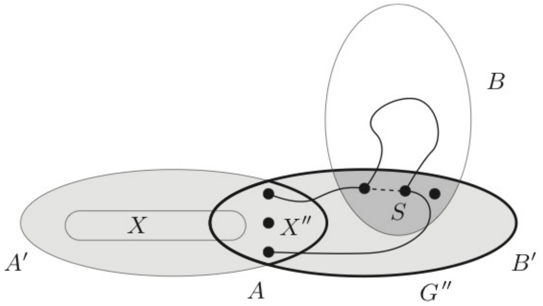

Fig. 3.5.3. If $S \subseteq B'$, then $(A', B' \cup B)$ is linked in G

By the induction hypothesis, therefore, X'' is linked in G''. But then X'' is also linked in $G[B' \cup B]$: as S was linked in $G[B]$, we simply replace any edges added on S in the definition of G' by disjoint paths through B (Fig. 3.5.3). But now $(A', B' \cup B)$ is a linked X-separation of G that violates the minimality of A in the choice of (A, B).

We have thus shown that G' satisfies the premise of $(*)$ with respect to X. Since $\{A, B\}$ is a proper separation, G' has fewer vertices than G. By the induction hypothesis, therefore, X is linked in G'. Replacing edges of $G' - E$ on S by paths through B as before, we can turn any linkage of X in G' into one in G, completing the proof of $(*)$. This completes the proof of (1).

$G[X]$

Our next goal is to show that, by the induction hypothesis, we may assume that G has not only large average degree but even large minimum degree. For our proof that X is linked in G, let $s_1, \dots, s_\ell, t_1, \dots, t_\ell$ be the distinct vertices in X which we wish to link by disjoint paths $P_i = s_i \dots t_i$. Let us add to G any missing edges on X except those of the form s_it_i; as the paths P_i are not allowed to have inner vertices in X, these new edges affect neither the premise nor the conclusion in $(*)$.

After this modification, we can now prove the following:

We may assume that any two adjacent vertices u, v which do not both lie in X have at least $8k-1$ common neighbours. (2)

$e = uv$
n
G'
V'

To prove (2), let $e = uv$ be such an edge, let n denote the number of common neighbours of u and v, and let $G' := G/e$ be the graph obtained by contracting e. Since u, v are not both in X we may view X as a subset also of $V' := V(G')$, replacing u or v in X with the contracted vertex v_e if $X \cap \{u, v\} \neq \emptyset$. Our aim is to show that unless $n \geqslant 8k-1$ as desired in (2), G' satisfies the premise of $(*)$. Then X will be linked in G' by the induction hypothesis, so the desired paths $P_1, \dots, P_\ell$ exist in G'. If one of them contains v_e, replacing v_e by u or v or uv turns it into a path in G, completing the proof of $(*)$.

In order to show that G' satisfies the premise of $(*)$ with respect to X, let us show first that $V' \setminus X$ is heavy. Since $V \setminus X$ was heavy and $|V' \setminus X| = |V \setminus X| - 1$, it suffices to show that the contraction of e resulted in the loss of at most $8k$ edges incident with a vertex outside X. If u and v are both outside X, then the number of such edges lost is only $n+1$: one edge at every common neighbour of u and v, as well as e. But if $u \in X$, then $v \notin X$, and we lost all the X–v edges xv of G with $x \neq u$, too: while xv counted towards $\|V \setminus X\|^+$, the edge xv_e lies in $G'[X]$ and does not count towards $\|V' \setminus X\|^+$. If x is not a common neighbour of u and v, then this is an additional loss. But u is adjacent to every $x \in X \setminus \{u\}$ except at most one (by our assumption about $G[X]$), so every such x except at most one is in fact a common neighbour of u and v. Thus in total, we lost at most $n+2$ edges. Unless $n \geqslant 8k-1$ (which would prove (2) directly for u and v), this means that we lost at most $8k$ edges, as desired for our proof that $V' \setminus X$ is heavy.

(A', B')
X'
(A, B), X''

It remains to show that for every small X-separation (A', B') of G' the set $B' \setminus A'$ is light. Let (A', B') be a counterexample, chosen with B' minimal. Then $G'[B']$, as in the proof of (1), satisfies the premise of $(*)$ with respect to $X' := A' \cap B'$. Hence X' is linked in $G'[B']$ by induction. Let A and B be obtained from A' and B' by replacing v_e, where applicable, with both u and v, and put $X'' := A \cap B$. We shall prove that the separation (A, B) of G contradicts our premise in $(*)$ or our assumption (1). Let us consider the three possible positions of v_e in turn: that v_e lies in $A' \smallsetminus B'$, in $B' \smallsetminus A'$, or in X'.

If $v_e \in A' \smallsetminus B'$, then $u, v \in A \smallsetminus B$. Then (A, B) is a small X-separation of G with $B \smallsetminus A = B' \smallsetminus A'$ heavy, contradicting our assumptions in $(*)$.

If $v_e \in B' \smallsetminus A'$ then $u, v \in B \smallsetminus A$. Now $X'' = X'$ is linked in $G[B]$, because it is linked in $G'[B']$: if v_e occurs on one of the linking paths for X', just replace it by u or v or uv as earlier. This contradicts (1).

Finally, assume that $v_e \in X'$. We show that $G[B]$ satisfies the premise of $(*)$ with respect to X''; then X'' will be linked in $G[B]$ by induction, again contradicting (1). Since (A', B') is a small X-separation, we have

$$|X''| \leqslant |X'| + 1 \leqslant |X| \leqslant 2k\,.$$

Moreover, $B \smallsetminus X'' = B' \smallsetminus A'$ is heavy in G, because it is heavy in G' by the choice of (A', B'). Now consider a small X''-separation (A'', B'') of $G[B]$. Then $(A \cup A'', B'')$ is a small X-separation of G, because $|X''| \leqslant |X|$. Therefore $B'' \smallsetminus A'' = B'' \smallsetminus (A \cup A'')$ is light by the assumption in $(*)$. Hence $G[B]$ does satisfy the premise of $(*)$ for X'', completing the proof of (2).

Using induction by contracting an edge, we have just shown that the vertices in $V \smallsetminus X$ may be assumed to have large degree. Using induction by deleting an edge, we now show that their degrees cannot be too large. Since $(*)$ holds if $V = X$, we may assume that $V \smallsetminus X \neq \emptyset$; let d^* denote d^*
the smallest degree in G of a vertex in $V \smallsetminus X$. Let us prove that

$$8k \leqslant d^* \leqslant 16k - 1\,. \tag{3}$$

The lower bound in (3) follows from (2) if we assume that G has no isolated vertex outside X, which we may clearly assume by induction. To prove the upper bound, let us see what happens if we delete an edge
e whose ends u, v are not both in X. If $G - e$ satisfies the premise of $(*)$ $e = uv$
with respect to X, then X is linked in $G - e$ by induction, and hence in G. If not, then either $V \smallsetminus X$ is light in $G - e$, or $G - e$ has a small X-separation (A, B) such that $B \smallsetminus A$ is heavy. If the latter happens then e must be an $(A \smallsetminus B)$–$(B \smallsetminus A)$ edge: otherwise, (A, B) would be a small X-separation also of G, and $B \smallsetminus A$ would be heavy also in G, in contradiction to our assumptions in $(*)$. But if e is such an edge then any common neighbours of u and v lie in $A \cap B$, so there are fewer than $|X| \leqslant 2k$ such neighbours. This contradicts (2).

So $V \smallsetminus X$ must be light in $G - e$. For G, this yields

$$\|V \smallsetminus X\|^+ \leqslant 8k\,|V \smallsetminus X| + 1\,. \tag{4}$$

$f(x)$

In order to show that this implies the desired upper bound for d^*, let us estimate the number $f(x)$ of edges that a vertex $x \in X$ sends to $V \setminus X$. There must be at least one such edge, xy say, as otherwise $(X, V \setminus \{x\})$ would be a small X-separation of G that contradicts our assumptions in $(*)$. But then, by (2), x and y have at least $8k-1$ common neighbours, at most $2k-1$ of which lie in X. Hence $f(x) \geqslant 6k$. As

$$2\,\|V \setminus X\|^+ = \sum_{v \in V \setminus X} d_G(v) + \sum_{x \in X} f(x)\,,$$

an assumption of $d^* \geqslant 16k$ would thus imply that

$$2\,(8k\,|V \setminus X| + 1) \underset{(4)}{\geqslant} 2\,\|V \setminus X\|^+ \geqslant 16k\,|V \setminus X| + 6k\,|X|\,,$$

yielding the contradiction of $2 \geqslant 6k\,|X|$. This completes the proof of (3).

To complete our proof of $(*)$, pick a vertex $v_0 \in V \setminus X$ of degree d^*, and consider the subgraph H induced in G by v_0 and its neighbours. By (2) we have $\delta(H) \geqslant 8k$, and by (3) and the choice of v_0 we have $|H| \leqslant 16k$. By Lemma 3.5.4, then, H has a k-linked subgraph; let L be its vertex set. By definition of 'k-linked', we have $|L| \geqslant 2k \geqslant |X|$. If G contains $|X|$ disjoint X–L paths, then X is linked in G, as desired. If not, then G has a small X-separation (A, B) with $L \subseteq B$. If we choose (A, B) of minimum order, then $G[B]$ contains $|A \cap B|$ disjoint $(A \cap B)$–L paths by Menger's theorem (3.3.1). But then (A, B) is a linked X-separation that contradicts (1). □

Exercises

For the first three exercises let G be a graph with vertices a and b, and let $X \subseteq V(G) \setminus \{a, b\}$ be an a–b separator in G.

1.$^-$ Show that X is minimal as an a–b separator if and only if every vertex in X has a neighbour in the component C_a of $G - X$ containing a, and another in the component C_b of $G - X$ containing b.

2.$^-$ Let $X' \subseteq V(G) \setminus \{a, b\}$ be another a–b separator, and define C'_a and C'_b correspondingly. Show that both

$$Y_a := (X \cap C'_a) \cup (X \cap X') \cup (X' \cap C_a)$$

and

$$Y_b := (X \cap C'_b) \cup (X \cap X') \cup (X' \cap C_b)$$

separate a from b (Figure 3.6.1).

3. Are Y_a and Y_b minimal a–b separators if X and X' are? Are $|Y_a|$ and $|Y_b|$ minimal for a–b separators from $V(G) \setminus \{a, b\}$ if $|X|$ and $|X'|$ are?

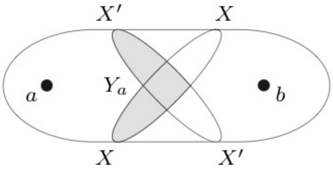

Fig. 3.6.1. The separators in Exercise 2

4. Let X and X' be minimal separators in G such that X meets at least two components of $G - X'$. Show that X' meets all the components of $G - X$, and that X meets all the components of $G - X'$.

5.$^-$ Deduce the $k = 2$ case of Menger's theorem (3.3.1) from Proposition 3.1.1.

6.$^-$ Prove the elementary properties of blocks mentioned after their formal definition.

7. Show that the block graph of any connected graph is a tree.

8.$^-$ Let G be a k-connected graph, and let xy be an edge of G. Show that G/xy is k-connected if and only if $G - \{x, y\}$ is $(k-1)$-connected.

9. Let e be an edge in a 2-connected graph $G \neq K^3$. Show that either $G - e$ or G/e is again 2-connected. Deduce a constructive characterization of the 2-connected graphs analogous to Theorems 3.2.3 and 3.2.5.

10.$^+$ Let e be an edge in a 3-connected graph $G \neq K^4$. Show that either $G \dot{-} e$ or G/e is again 3-connected.

11. Show without using Theorem 3.2.6 that every edge of a 3-connected graph lies on some non-separating induced cycle.

12. Give an inductive proof of Theorem 3.2.6 based on Lemma 3.2.2. You may use the previous exercise.

13.$^+$ Give an inductive proof of Theorem 3.2.6 based on Lemma 3.2.4.

14.$^+$ Show that every transitive graph G with $\kappa(G) = 2$ is a cycle.

15. Find the error in the following 'simple proof' of Menger's theorem (3.3.1). Let X be an A–B separator of minimum size. Denote by G_A the subgraph of G induced by X and all the components of $G - X$ that meet A, and define G_B correspondingly. By the minimality of X, there can be no A–X separator in G_A with fewer than $|X|$ vertices, so G_A contains k disjoint A–X paths by induction. Similarly, G_B contains k disjoint X–B paths. Together, all these paths form the desired A–B paths in G.

16. Prove Menger's theorem by induction on $\|G\|$, as follows. Given an edge $e = xy$, consider a smallest A–B separator S in $G - e$. Show that the induction hypothesis implies a solution for G unless $S \cup \{x\}$ and $S \cup \{y\}$ are smallest A–B separators in G. Then show that if choosing neither of these separators as X in the previous exercise gives a valid proof, there is only one easy case left to do.

17. Work out the details of the proof of Corollary 3.3.5 (ii).

18. Let $k \geqslant 2$. Show that every k-connected graph of order at least $2k$ contains a cycle of length at least $2k$.

19. Let $k \geqslant 2$. Show that in a k-connected graph any k vertices lie on a common cycle.

20. Find a subset D of the plane and two infinite subsets $A, B \subseteq D$ such that for every finite set $X \subseteq D$ there is an A–B arc in $D \smallsetminus X$ but D contains no infinite set of disjoint A–B arcs.

21. Derive the edge part of Corollary 3.4.2 from the vertex part.

    (Hint. Consider the H-paths in the graph obtained from the disjoint union of H and the line graph $L(G)$ by adding all the edges he such that h is a vertex of H and $e \in E(G) \smallsetminus E(H)$ is an edge at h.)

22.$^-$ To the disjoint union of the graph $H = \overline{K^{2m+1}}$ with k copies of K^{2m+1} add edges joining H bijectively to each of the K^{2m+1}. Show that the resulting graph G contains at most $km = \frac{1}{2}\kappa_G(H)$ independent H-paths.

23. Find a bipartite graph G, with partition classes A and B say, such that for $H := G[A]$ there are at most $\frac{1}{2}\lambda_G(H)$ edge-disjoint H-paths in G.

24.$^+$ Derive Tutte's 1-factor theorem (2.2.1) from Mader's theorem.

    (Hint. Extend the given graph G to a graph G' by adding, for each vertex $v \in G$, a new vertex v' and joining v' to v. Choose $H \subseteq G'$ so that the 1-factors in G correspond to the large enough sets of independent H-paths in G'.)

25.$^-$ Show that k-linked graphs are $(2k-1)$-connected. Are they even $2k$-connected?

26. For every $k \in \mathbb{N}$ find an $\ell = \ell(k)$, as large as possible, such that not every ℓ-connected graph is k-linked.

27. Show that if G is k-linked and $s_1, \dots, s_k, t_1, \dots, t_k$ are not necessarily distinct vertices such that $s_i \neq t_i$ for all i, then G contains independent paths $P_i = s_i \dots t_i$ for $i = 1, \dots, k$.

28. Go through the proof of Theorem 3.5.3 monitoring the use of $\|V \smallsetminus X\|^+$. How would the proof fail if $\|G[V \smallsetminus X]\|$ was used instead? Which arguments would become simpler?

29. Use Theorem 3.5.3 to show that the function h in Lemma 3.5.1 can be chosen as $h(r) = cr^2$, for some $c \in \mathbb{N}$.

Notes

Although connectivity theorems are doubtless among the most natural, and also the most applicable, results in graph theory, there is still no monograph on this subject. The most comprehensive source is perhaps A. Schrijver, *Combinatorial optimization*, Springer 2003, together with a number of surveys on specific topics by A. Frank, to be found on his home page. Some areas are covered in B. Bollobás, *Extremal Graph Theory*, Academic Press 1978, in R. Halin, *Graphentheorie*, Wissenschaftliche Buchgesellschaft 1980, and in A. Frank's chapter of the *Handbook of Combinatorics* (R.L. Graham, M. Grötschel & L. Lovász, eds.), North-Holland 1995. A survey specifically of techniques and results on minimally k-connected graphs (see below) is given by W. Mader, On vertices of degree n in minimally n-connected graphs and digraphs, in (D. Miklós, V.T. Sós & T. Szőnyi, eds.) *Paul Erdős is 80*, Vol. 2, Proc. Colloq. Math. Soc. János Bolyai, Budapest 1996.

Theorem 3.2.3 is due to W.T. Tutte, *Connectivity in graphs*, Oxford University Press 1966. Tutte's *wheel theorem*, proved in W.T. Tutte, A theory of 3-connected graphs, *Nederl. Akad. Wet. Proc. Ser. A* **64** (1961), 441–455, differs from our Theorem 3.2.5 as follows. As an alternative to the contraction of an edge in the reduction step, the wheel theorem also allows its deletion. Of edges to be contracted, however, it requires that they do not lie in any triangle. The starting set for the construction of all 3-connected graphs therefore consists of all wheels, not only K^4. Tutte's wheel theorem has been extended to 3-connected graphs H other than K^4: from any 3-connected graph $G \succcurlyeq H$ that is not a wheel we can obtain H by contracting or deleting edges step by step, remaining 3-connected at every step. (As in Tutte's theorem, one is not allowed to contract edges that lie in a triangle.) This was proved by S. Negami, A characterization of 3-connected graphs containing a given graph, *J. Comb. Theory B* **32** (1982), 69–74. It also follows from an earlier theorem of Seymour on matroid decompositions, and is sometimes called Seymour's *splitter theorem* for 3-connected graphs.

Our proof of Theorem 3.2.6 is the original from W.T. Tutte, How to draw a graph, *Proc. Lond. Math. Soc.* **13** (1963), 743–767. Alternative proofs are indicated in Exercises 12 and 13.

An approach to the study of connectivity not touched upon in this chapter is the investigation of *minimal* k-connected graphs, those that lose their k-connectedness as soon as we delete an edge. Like all k-connected graphs, these have minimum degree at least k, and by a fundamental result of Halin (1969), their minimum degree is exactly k. The existence of a vertex of small degree can be particularly useful in induction proofs about k-connected graphs. Halin's theorem was the starting point for a series of more and more sophisticated studies of minimal k-connected graphs; see the books of Bollobás and Halin cited above, and in particular Mader's survey.

Menger's theorem is probably the most-used classical result in graph theory. Our first proof is extracted from Halin's book. The second is due to T. Böhme, F. Göring and J. Harant, Menger's theorem, *J. Graph Theory* **37** (2001), 35–36, the third to T. Grünwald (later Gallai), Ein neuer Beweis eines Mengerschen Satzes, *J. Lond. Math. Soc.* **13** (1938), 188–192. A fourth proof is sketched in Exercise 16, and in Chapter 6 we shall obtain a fifth proof as an application of a theorem about network flows (Ch. 6, Ex. 3.) The global version of Menger's theorem (Theorem 3.3.6) was first stated and proved by Whitney (1932).

Mader's Theorem 3.4.1 is taken from W. Mader, Über die Maximalzahl kreuzungsfreier H-Wege, *Arch. Math.* **31** (1978), 387–402; a short proof has been given by A. Schrijver, A short proof of Mader's $\mathcal{S}$-paths theorem, *J. Comb. Theory B* **82** (2001), 319–321. The theorem may be viewed as a common generalization of Menger's theorem and Tutte's 1-factor theorem (Exercise 24).

Theorem 3.5.3 is due to R. Thomas and P. Wollan, An improved linear bound for graph linkages, *Eur. J. Comb.* **26** (2005), 309–324. Using a more involved version of Lemma 3.5.4, they prove that $2k$-connected graphs even with only $\varepsilon \geqslant 5k$ must be k-linked. And for graphs of large enough girth the condition on ε can be dropped altogether: as shown by W. Mader, Topological subgraphs in graphs of large girth, *Combinatorica* **18** (1998), 405–412, such graphs are k-linked as soon as they are $2k$-connected, which is best possible. (Mader assumes a lower bound on the girth that depends on k, but this is not necessary; see D. Kühn & D. Osthus, Topological minors in graphs of large girth, *J. Comb. Theory B* **86** (2002), 364–380.) In fact, for every $s \in \mathbb{N}$ there exists a k_s such that if $G \not\supseteq K_{s,s}$ and $\kappa(G) \geqslant 2k \geqslant k_s$ then G is k-linked; see D. Kühn & D. Osthus, Complete minors in $K_{s,s}$-free graphs, *Combinatorica* **25** (2005) 49–64.