CSL758 Lecture Scribe (Date:24/4/2008)

Maximum Weight Bipartite Matching in an undirected graph

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1 Introduction

Two lectures back we studied the Hungarian Algorithm for finding the maximum weight bipartite matching in an undirected graph.In this lecture,we will study that for this problem,optimal solution to the integer program IP=optimal solution to the linear program LP(with relaxed constraints).

2 Weighted matching problem

The problem of finding the maximum weight matching in a bipartite graph can be formulated as an integer program in the following manner:-
Let x_e be a variable associated with the edge $e \in E$ such that

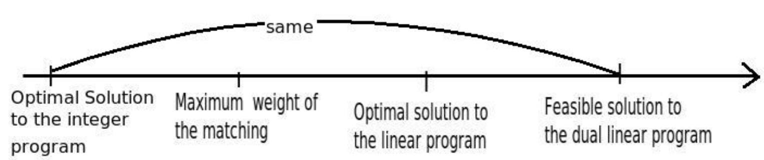
$x_e = 1$ if e is a matched edge,0 otherwise.

Thus we have to find $\max(\sum_e w_e * x_e)$ subject to the constraint that :-

$$\forall u \in U \cup V, \sum_{e \in \delta(v)} x_e <= 1$$

where U and V are the set of vertices as defined by Hungarian Algorithm. The optimal solution to this integer program is the maximum weight bipartite matching in the graph but this problem is NP-hard.So we have to convert this into a problem of linear programming.The linear programming formulation is through relaxing the constraint on x_e for each edge to $0 \leq x_e \leq 1$.

For this problem,from the Hungarian Algorithm,we have:-



where as we move right on the line,the solution becomes worse(that is,the total weight decreases).

Thus for maximum weighted bipartite matching problem,optimal solution to IP = optimal solution to LP.That is, a polytope for IP will also have integral vertices. In the next section,the reason for this is discussed.

3 Reason Behind Exact Relaxation

Consider the IP problem.
Consider an m-dimensional polytope where m is the no. of edges in the graph.
No. of constraints=number of vertices.
$2*n$(=no. of vertices) intersections are taken to form $m-2n-1=k$ dimensional hyperplane.
Thus we need an additional k hyperplanes to define a vertex from $x_e >= 0$ which when satisfied an equality would give me a vertex.
So out of $2n+m$ hyperplanes,we have to pick k to give a vertex.We are setting these k to 0.
So in the adjacency matrix for the graph,we have an additional m rows one for each edge,k out of them set to 0 and k columns are set to zero(where columns are edges).
So the no. of columns=$m-(m-(2n-1)) = 2n-1$ and we get a $(2n-1)*(2n-1)$ square matrix(A) and we have to solve $AX = B$ (where B is a column vector of 1's).
The corresponding LP problem will also have the same matrix A and same problem $AX = B$.
We consider the LP problem.To prove the integer solution to $AX = B$,we show that $det(A)$=1 or -1 or 0.
We expand A along the column which has one 1,we eventually get $det(A)$=1 or -1 or 0.