A *cycle cover* in a directed graph *G=(V,E)* is a subset of vertices, $V' \subset V$, such that every cycle of *G* passes through a vertex in V'. Given a directed graph *G* and an integer *k* the *cycle cover problem* is to determine if there is a cycle cover of size at most *k*. Show that this problem is NP-complete by reducing the vertex-cover problem to it.(10)

To prove that the problem is in NP.

- a certificate for ~~the~~ a 'YES' instance is a subset, say S, of at most k vertices which forms a cycle cover [2 marks]
- the certifier needs to check in polynomial time that S is indeed a cycle cover. This can be done by removing S & incident edges & checking that the remaining graph is acyclic (using DFS as done in class). [2 marks]

To prove the problem is NP-hard.

Let $(G=(V,E), k)$ be an instance of the vertex cover problem. Construct $G'=(V,E')$ by replacing each edge $e=(u,v) \in E$ with two directed edges $e_1=(u,v)$ & $e_2=(v,u)$ [2 marks]

Claim: G' has a cycle cover of size k iff G has a vertex cover of size k.

Proof. G' has cyclecover of size $k \Rightarrow G$ has vertex cover of size k.

Let S be a cycle cover in G' having k vertices. In G' whenever we have an edge (u,v) we also have edge (v,u). Hence there is no edge between any two vertices in $V-S$ because then there would also be the reverse edge & therefore a cycle. So $V-S$ is an independent set in G & hence S is a vertex cover in G [2 marks]

G has a vertex cover of size $k \Rightarrow G'$ has a cycle cover of size k

Let S be a vertex cover in G. Since $V-S$ is independent in G, in G' there is no edge between any two vertices in $V-S$. Hence S is a cycle cover in G'. [2 marks]

COMMON MISTAKE

Reducing cycle cover to vertex cover.