Approximate distances in weighted graphs

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(Joint work with Surender Baswana)

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- $\hat{\delta}(u,v)$ is a t-stretch estimate of $\delta(u,v)$ if

$$\delta(u,v) \le \hat{\delta}(u,v) \le t\delta(u,v)$$

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$$O(\min(n^2 \log n, m\sqrt{n}))$$
- however, our query answering time is $O(\log n)$.

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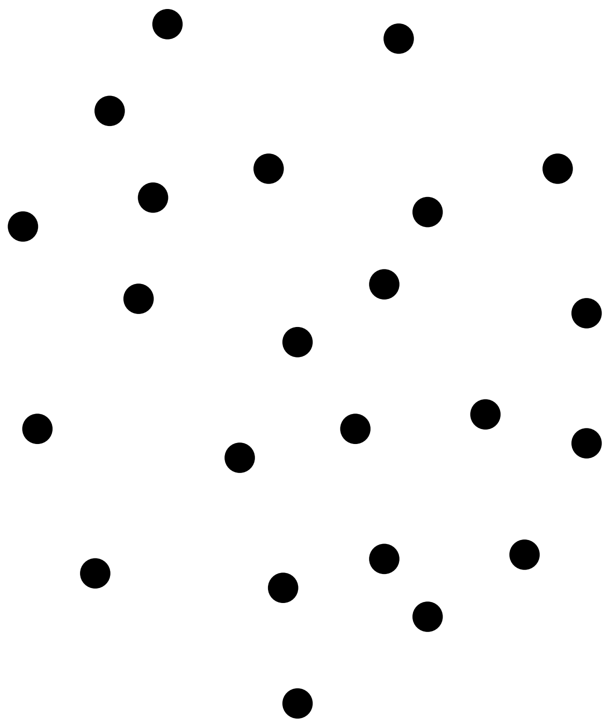
The Thorup-Zwick algorithm



- sample each vertex indep. with prob. $1/\sqrt{n}$.

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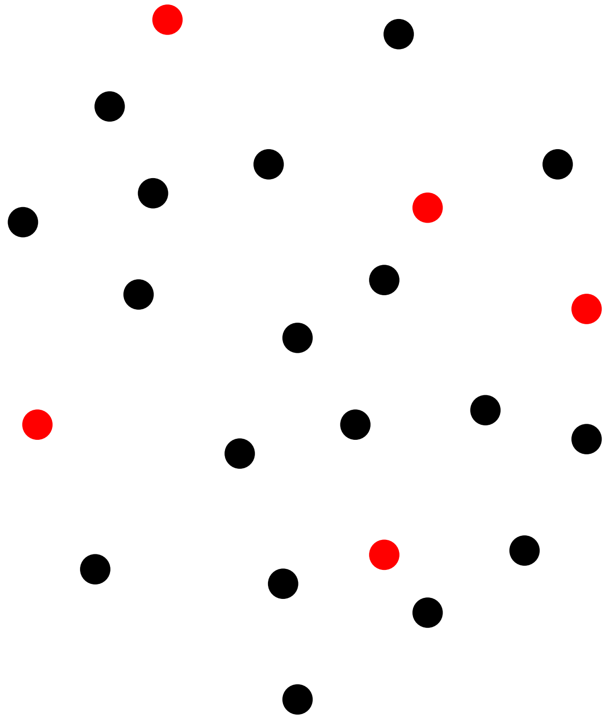

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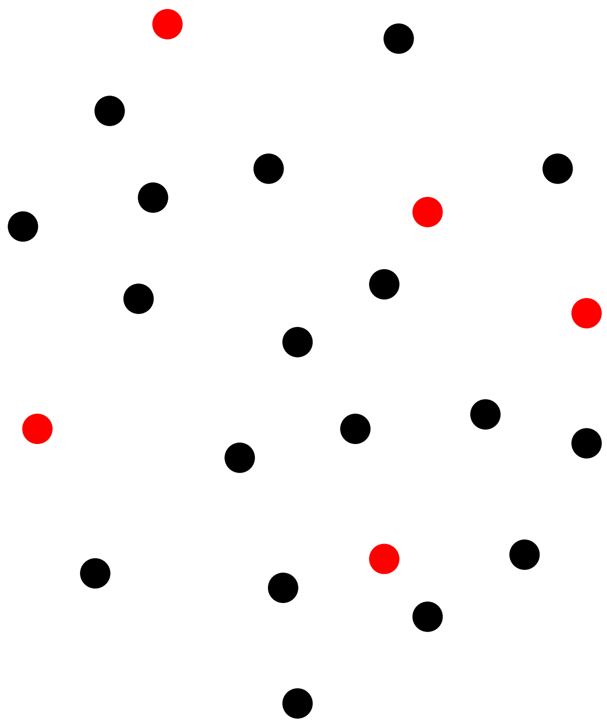

- let S be the set of sampled vertices.

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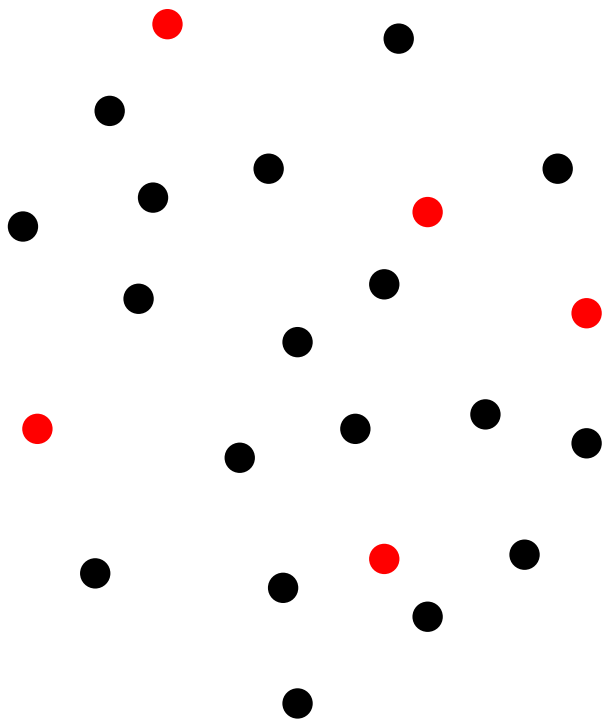

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- $E[|S|] = \sqrt{n}$.

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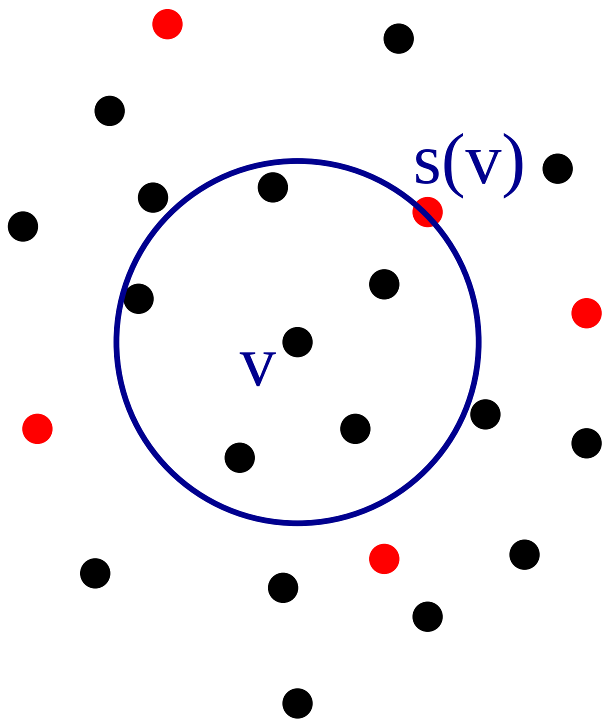

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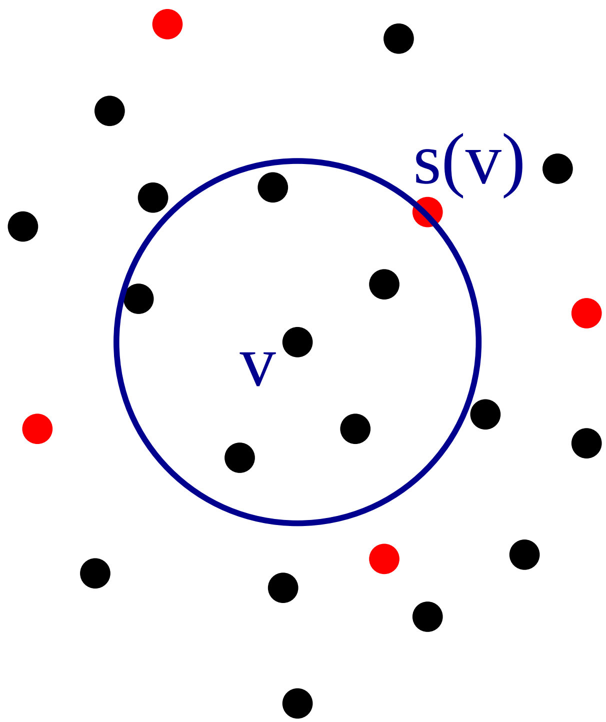

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* the query answering time is $O(1)$.

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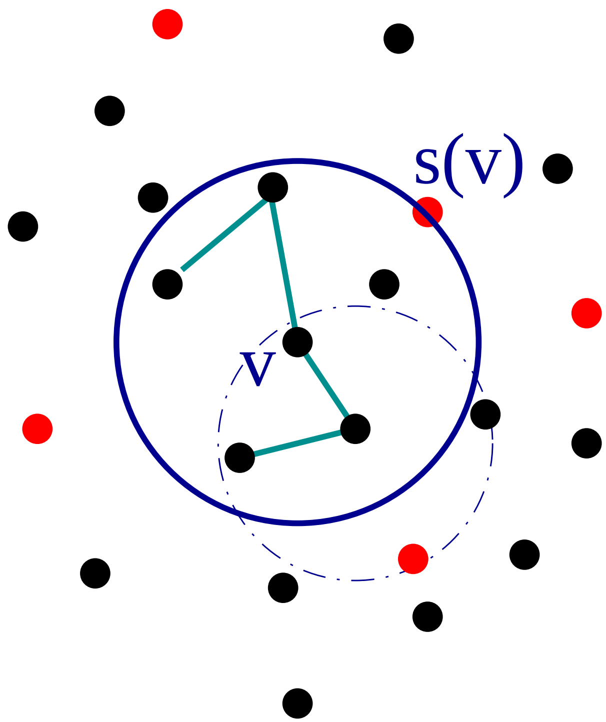

– *improving step 1*: each v performs truncated Dijkstra in a subgraph G' of G.

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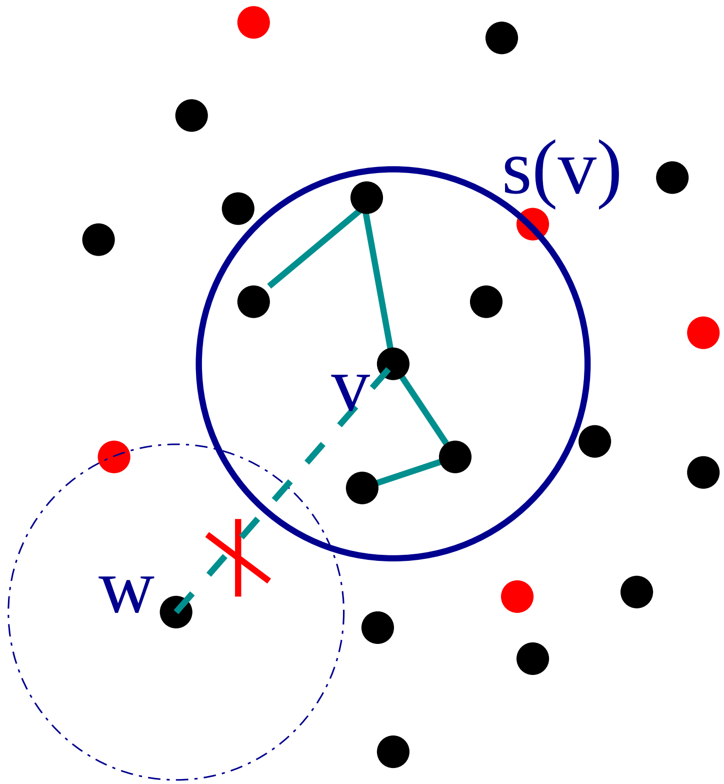

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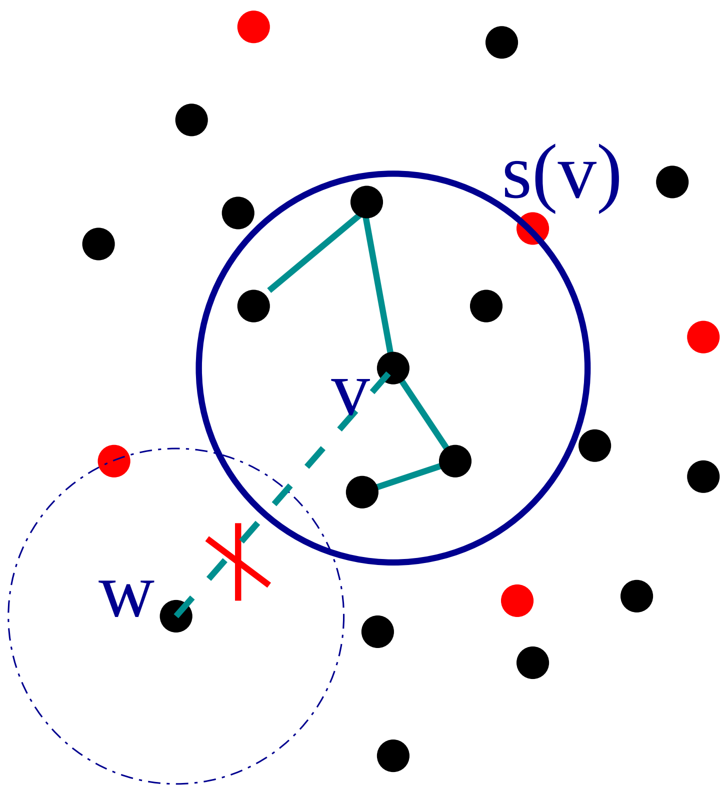

- an edge like (v, w) cannot lie in $b(u)$, for any u.
- G' contains no such edge: $E[|G'|] = n^{3/2}$.

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– *we now need to improve step 2:*

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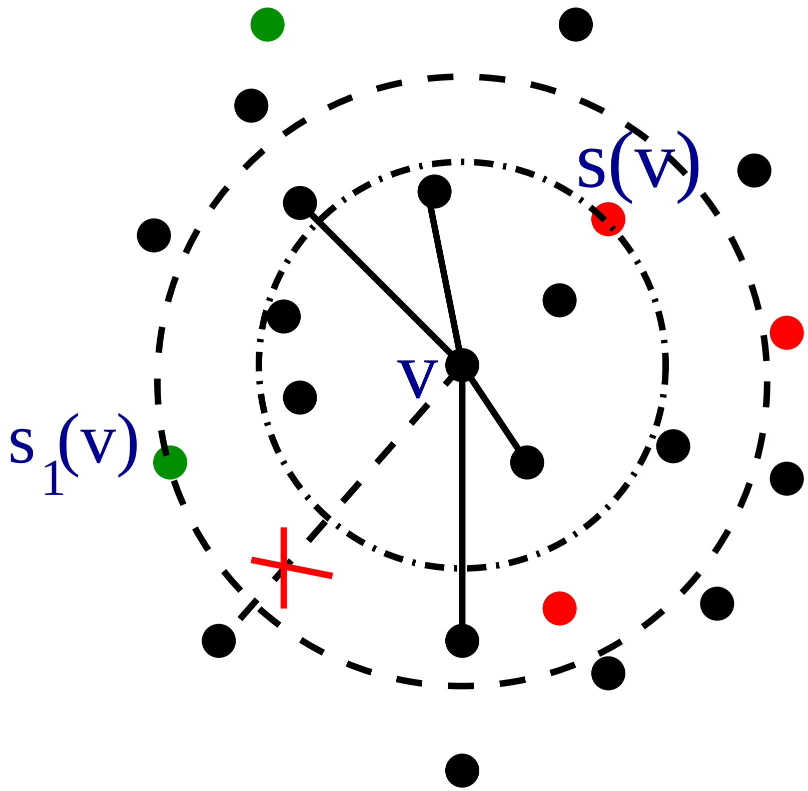

$G_1 = \cup_v$ edges of the form above.

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$$\le \delta(v, u) + \delta(s_i(v), v) + \delta(v, u) \le 3\delta(v, u)$$

for some $s_i(v)$ which knows $\delta(s_i(v), u)$.

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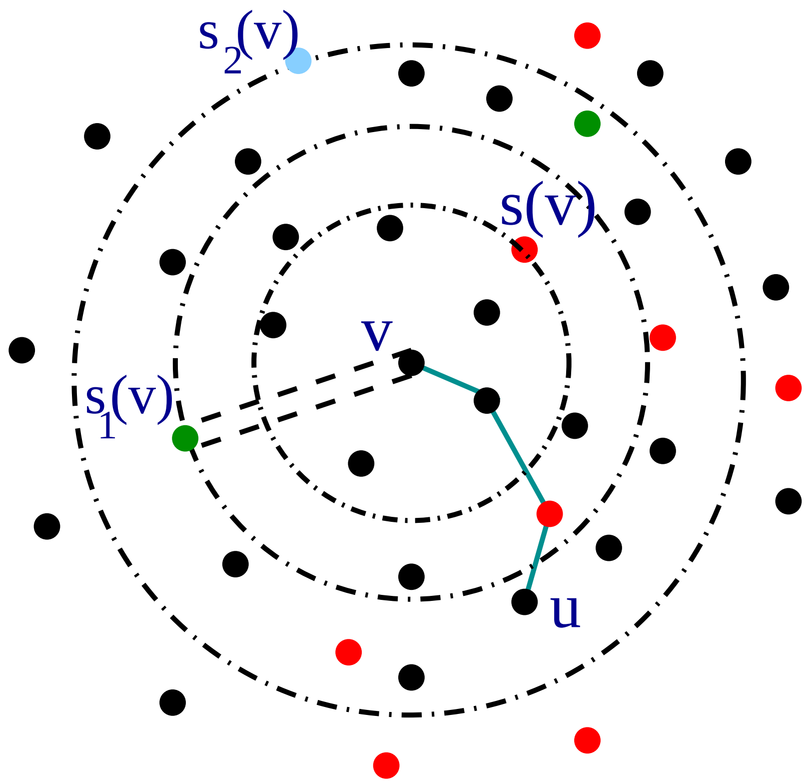

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- space requirement: $O(n^{3/2})$

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- each $s_i \in S_i$ does a Dijkstra in the subgraph G_{i+1}.
- G_{i+1}: consists of all edges incident on each v whose weight is less than $\delta(v, s_{i+1}(v))$.
- For each v, u in G

$$d[v,u] \leftarrow \min_{0 \le i \le \log n}(\delta(v, s_i(v)) + \hat{\delta}(u, s_i(v)), \delta(u, s_i(u)) + \hat{\delta}(v, s_i(u)))$$

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