CSL866: Percolation Theory and Random Graphs
I semester, 2007-08

Minor II

Due: In class on 29th October 2007

Examination notes: You are free to discuss the examination with other people but please mention the names of everyone you have discussed the paper with. Also, please write your own solutions out. For this exam, latexing the solutions is absolutely necessary. If you do not latex your solutions, your paper will simply not be accepted and it will not be graded.

Percolation on butterflies

In this exam we will think about percolation on the butterfly network.

A *butterfly* is a graph defined on an index set $I \subseteq \mathbb{Z}$ as follows:

- The vertex set $V = I \times 2^I$.
- A vertex $v = (i, S_i)$ has an edge to another vertex $u = (j, S_j)$ if
  1. $|i - j| = 1$ and $S_i = S_j$.
  2. $i - j = 1$ and $S_i \setminus S_j = \{j\}$ or $S_j \setminus S_i = \{j\}$.

See Figure 1 for an example of a butterfly for which $I = \{0, 1, 2, 3\}$. We define an infinite butterfly by setting $I = \mathbb{Z}$. For reference we call the vertex $w = (0, \emptyset)$ the origin of this infinite butterfly. As for $\mathbb{L}^2$ we define a percolation process on the infinite butterfly with each edge being open with probability p.

Now, we conjecture that this infinite butterfly has a non-trivial critical probability (this corresponds to Thm 2.3 of Lecture 2).

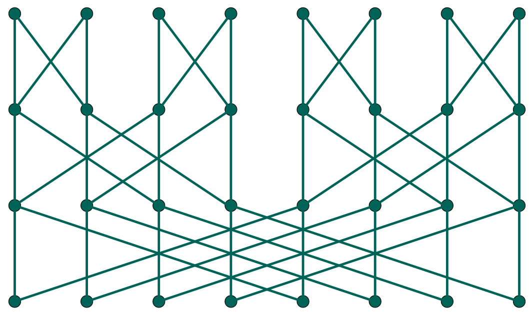

Figure 1: A butterfly with 4 levels.

Conjecture 2.1 *For the infinite butterfly there are constants α and β such that*

$$\alpha < p_c < \beta.$$

As part of the exam, try to prove this conjecture.

The first inequality should be easy to prove but for the second one, a good definition similar to the dual for the planar mesh is required.

Further, we attempt to define something similar to $B(n)$ for this setting.

$$BB(n) = \{(i, S) | -n \leq i \leq n, S \subseteq \{-n, -(n-1), \ldots, 0, \ldots, n-1, n\}\}.$$

As part of the exam, give a description in words and in mathematical notation of what the translated box $BB(n, x)$ will be.

Further, we conjecture that it should be possible to prove a theorem similar to Theorem 4.3 of lecture 4. Try to prove this theorem as part of the exam.

Can you also prove a lower bound similar to the one proved in Lecture 8?