Problem 1.1 (60=10+10+10+10+10+10 marks) Attempt problems 1 to 6 of Sec 1.9 of [Spielman2015].

Problem 1.2 (10 marks) Construct a non-symmetric matrix \( M \) with two distinct eigen values \( \mu \) and \( \nu \) such that
\[
M \psi = \mu \psi \quad \text{and} \quad M \phi = \nu \phi
\]
But \( \psi \) and \( \phi \) are not orthogonal, i.e.,
\[
\psi^T \phi \neq 0
\]

Problem 1.3 (30 marks) Let \( M \) be an \( n \times n \) real symmetric matrix with largest eigen value \( \lambda \) and corresponding eigen vector \( x \). Assume that \( x \) maximises the Rayleigh quotient
\[
\frac{x^T M x}{x^T x}
\]
Extend the proof of this fact given in Spielman’s notes to prove the Spectral Theorem for real symmetric matrices, i.e., there exist real numbers \( \lambda_1, \ldots, \lambda_n \) and \( n \) mutually orthogonal unit vectors \( \psi_1, \ldots, \psi_n \) and such that \( \psi_i \) is an eigenvector of \( M \) of eigenvalue \( \lambda_i \), for each \( i, 1 \leq i \leq n \).