Important: The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

Problem 1 [1, Prob 12, page 30]
Show that every 2-connected graph contains a cycle.

Problem 2
Show using only the material covered in [1, Ch 1.4] that every connected graph on \( n \) vertices has at least \( n - 1 \) edges.

Problem 3
Generalize the result of Problem 2 to show that every graph on \( n \) vertices and \( m \) edges has at least \( n - m \) components.

Problem 4 ♠
Given a graph \( G = (V, E) \) and a minimal cut \( F \subseteq E \), show that any cycle of \( G \) contains an even number of edges of \( F \) (this number could be 0 as well).

Problem 5
Show that if there is a vertex \( v \) of odd degree in graph \( G \) there must be a path from \( v \) to another vertex \( u \) of \( G \) which also has odd degree.

Problem 6
Let \( \bar{G} \) be the complement of the graph \( G \), i.e., all edges of \( G \) are non-edges of \( \bar{G} \) and vice versa. Show that both \( G \) and \( \bar{G} \) cannot be disconnected, i.e., at least one of them must be connected.

Problem 7
Given a graph \( G = (V, E) \) such that \( |V| = n \), a cut \( F \subseteq E \) is called a balanced cut if \( G \setminus F \) has exactly 2 components and each of these components has size at least \( n/3 \). Construct graphs on \( n \) vertices whose smallest balanced cut has size (a) \( \theta(1) \), (b) \( \theta(\sqrt{n}) \) and (c) \( \theta(n) \).

Problem 8 (Menger’s Theorem)
Prove that a graph \( G \) has \( \lambda(G) = k \) for any \( k \geq 1 \) iff there are \( k \) edge-disjoint paths between any pair of vertices in \( G \). Two paths are said to be independent if they don’t share any edges. Caution: One direction of this theorem is easy and the other is tricky.

References