COL202: Discrete Mathematical Structures. I semester, 2020-21.
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Tutorial Sheet 3: Basics of graphs.
15 October 2020

Important: The question marked with a ♠ is to be submitted via gradescope by 11:59PM on the day that you have your tutorial.

Note: This sheet contains a problem marked with a (*). This is a somewhat open-ended problem for independent work on your own.

Problem 1
Prove that every simple graph has two vertices of the same degree.

Problem 2 [1, Prob 2, page 30]
Let $d \in \mathbb{N}$ and $V = \{0,1\}^d$, i.e., V is the set of all 0-1 sequences of length d. We define the edge set as follows: there is an edge between two sequences if they differ in exactly one position. This graph is known as the d-dimensional cube. Determine the average degree, diameter, girth and circumference of the *d-dimensional cube*. Note that the circumference of a graph is the length of the longest cycle in the graph.

Problem 3 [1, Prob 3, page 30]
Let G be a graph containing a cycle C, and assume that G contains a path of length at least k between two vertices of C. Show that G contains a cycle of length at least $\sqrt{k}$.

Problem 4
Given a graph $G = (V, E)$, a sorted ascending sequence made from the number $\{d(v) : v \in V\}$ is call the *degree sequence* of the graph. Clearly if two graphs are isomorphic their degree sequence is the same. Construct a counterexample to show that the converse is not true.

Problem 5 [1, Prob 6, page 30]
Show that $\text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G)$ for every graph G.

Problem 6 [1, Prob 7, page 30]
For $d \in \mathbb{R}$ and $g \in \mathbb{N}$, define

$$n_0(d, g) = 1 + d \sum_{i=0}^{r-1} (d-1)^i,$$

if $g = 2r + 1$ is odd and

$$n_0(d, g) = 2 \sum_{i=0}^{r-1} (d-1)^i,$$

if $g = 2r$ is even. Show that a graph with minimum degree δ and girth g has at least $n_0(\delta/2, g)$ vertices. You can assume $\delta \geq 2$.

Problem 7
Given a set X, a function $f : X \times X \to [0, \infty)$ is called a *distance* if

1. $\forall x, y \in X : f(x, y) = 0 \Leftrightarrow x = y$,
2. $\forall x, y \in X : f(x, y) = f(y, x)$, and

3. $\forall x, y, z \in X : f(x, y) \leq f(x, z) + f(z, y)$.

Problem 7.1
Prove that the graph distance defined as the length of the shortest path between two vertices is a distance.

Problem 7.2
Suppose that given a graph $G = (V, E)$ we have a function $w : E \to \mathbb{R}$ and we define the length of the path $x_0 \dots x_k$ to be $\sum_{i=1}^{k} w(x_{i-1}x_i)$. As before we define the "distance" between two vertices to be the length of the shortest path between the two vertices. What condition do we need on w for this "distance" to actually be a distance? Which of the requirements of a distance get violated if w is allowed to assign negative values? Do any requirements get violated if w is allowed to assign the value 0?

Problem 8 ♠
Given two graphs $G = (V, E)$ and $G' = (V', E')$ such that $|V| = |V'|$, suppose we can find a $\phi : V \to V'$ which is a bijection and is a graph homomorpishm. Prove that diameter$(G') \leq$ diameter(G). Draw an example where the inequality is strict. Use this fact to prove that two isomorphic graphs have the same diameter.

Problem 9 *
Given a set of vertices V such that $|V| = n$, and given k such that $\binom{n}{2} \geq k \geq 0$, let us denote by $A_{n,k}$ the set of all simple graphs on V with exactly k edges. We now define a graph whose vertices are the elements of $A_{n,k}$. We put an edge between graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ is $|E_1 \setminus E_2| = 1$. What is the diameter of this graph in terms of k? Does the diameter always increase as k increases?

References

[1] Reinhard Diestel, *Graph Theory 5ed.*, Springer, 2016.