Problem 1 [1, Prob 2, page 30]
Let \( d \in \mathbb{N} \) and \( V = \{0,1\}^d \), i.e., \( V \) is the set of all 0-1 sequences of length \( d \). We define the edge set as follows: there is an edge between two sequences if they differ in exactly one position. This graph is known as the \( d \)-dimensional cube. Determine the average degree, diameter, girth and circumference of the \( d \)-dimensional cube. Note that the circumference of a graph is the length of the longest cycle in the graph.

Problem 2 [1, Prob 3, page 30]
Let \( G \) be a graph containing a cycle \( C \), and assume that \( G \) contains a path of length at least \( k \) between two vertices of \( C \). Show that \( G \) contains a cycle of length at least \( \sqrt{k} \).

Problem 3 [1, Prob 6, page 30]
Show that \( \text{rad}(G) \leq \text{diam}(G) \leq 2\text{rad}(G) \) for every graph \( G \).

Problem 4 [1, Prob 7, page 30]
For \( d \in \mathbb{R} \) and \( g \in \mathbb{N} \), define
\[
n_0(d,g) = 1 + d \sum_{i=0}^{r-1} (d-1)^i,
\]
if \( g = 2r + 1 \) is odd and
\[
n_0(d,g) = 2 \sum_{i=0}^{r-1} (d-1)^i,
\]
if \( g = 2r \) is even. Show that a graph with minimum degree \( \delta \) and girth \( g \) has at least \( n_0(\delta/2, g) \) vertices. You can assume \( \delta \geq 2 \).

Problem 5 [1, Prob 12, page 30]
Show that every 2-connected graph contains a cycle.

Problem 6
Show that if there is a vertex \( v \) of odd degree in graph \( G \) there must be a path from \( v \) to another vertex \( u \) of \( G \) which also has odd degree.

Problem 7
Let \( G \) be the complement of the graph \( \bar{G} \), i.e., all edges of \( G \) are non-edges of \( \bar{G} \) and vice versa. Show that both \( G \) and \( \bar{G} \) cannot be disconnected, i.e., at least one of them must be connected.

References