COL 726

Homework VI

Due on May 4, 2018

1. **(Heath)** Are solutions of the following system of ODE's stable ? Give reasons.
$$y_1' = -y_1 + y_2, \; y_2' = -2y_2.$$

2. **(Heath)** The centered difference approximation
$$y' \approx \frac{y_{k+1} - y_{k-1}}{2h}$$
leads to the two step leapfrog method
$$y_{k+1} = y_{k-1} + 2hf(t_k, y_k)$$
for solving the ODE $y' = f(t, y)$. Determine the order of accuracy and the stability region of the method.

3. **(Heath)** Consider the system of ODEs :
$$y' = \begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} -k_1 y_1 \\ k_1 y_1 - k_2 y_2 \\ k_2 y_2 \end{bmatrix}.$$
Solve the system numerically assuming the initial conditions: $y_1(0) = y_2(0) = y_3(0) = 1$. Take $k_1 = 1$ and experiment with values of k_2 of varying magnitude, specifically $k_2 = 10, 100$ and 1000. For each value of k_2, solve the system using fourth order Runge-Kutta method and implicit Euler method. Compare the two methods with respect to accuracy and stability for a given step size. In each instance, integrate the ODE system from $t = 0$ until the solutin approaches a steady state or until the method is clearly unstable or grossly inefficient.

4. **(Heath)** The following system of ODEs, formulated by Lorentz, represents a crude model of atmospheric circulation:
$$\begin{aligned} y_1' &= \sigma(y_2 - y_1) \\ y_2' &= r y_1 - y_2 - y_2 y_3 \\ y_3' &= y_1 y_2 - y_3 \end{aligned}$$
Taking $\sigma = 10, b = 8/3, r = 28$ and intial values $y_1(0) = y_3(0) = 0$ and $y_2(0) = 1$, integrate this ODE from $t = 0$ to $t = 100$ (use Runge-Kutta fourth order method). Plot each of y_1, y_2 and y_3 as a function of t, and also plot each of the trajectories $(y_1(t), y_2(t))$, $(y_1(t), y_3(t))$ and $(y_2(t), y_3(t))$ as a function of t, each on a separate plot. Try perturbing the initial values by a tiny amount and see how much difference this makes in the final value of $y(100)$.

5. Consider the following implicit method iteration for solving the ODE $y' = f(t, y)$, where α is a parameter between 0 and 1:

$$y_{k+1} = y_k + h\left[(1-\alpha)f(t_k, y_k) + \alpha f(t_{k+1}, y_{k+1}\right].$$

Show that the local error is $O(h^2)$, unless θ is 1/2, in which case it is $O(h^3)$. You can assume that all high order derivatives of y are bounded, and f satisfies the following Lipschitz condition for all t, y, z (in its domain):

$$|f(t, y) - f(t, z)| \leq L|y - z|.$$

6. Show that if $x = x_0$ is a zero of $f(x)$ of multiplicity m (i.e., $f(x_0) = 0, f'(x_0) = 0, \ldots, f^{(m-1)'}(x_0) = 0, f^{(m)'}(x_0) \neq 0$), then the iteration

$$x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$$

converges quadratically under suitable continuity conditions.

7. **(Heath)**Consider the function $g(x) = x - x^3$. Find the unique fixed point of $g(x)$. Prove that the fixed-point iteration converges to this unique fixed point if the starting point lies in the open interval $(-1, 1)$. Is it true that, for some $k < 1$ and all n, $|e_n| \leq k|e_{n-1}|$?