1. (Heath) Are solutions of the following system of ODE’s stable? Give reasons.

\[ y_1' = -y_1 + y_2, \quad y_2' = -2y_2. \]

2. (Heath) The centered difference approximation

\[ y' \approx \frac{y_{k+1} - y_{k-1}}{2h} \]

leads to the two step leapfrog method

\[ y_{k+1} = y_{k-1} + 2hf(t_k, y_k) \]

for solving the ODE \( y' = f(t, y) \). Determine the order of accuracy and the stability region of the method.

3. (Heath) Consider the system of ODEs:

\[
\begin{bmatrix}
  y_1' \\
  y_2' \\
  y_3'
\end{bmatrix} = \begin{bmatrix}
  -k_1y_1 \\
  k_1y_1 - k_2y_2 \\
  k_2y_2
\end{bmatrix}.
\]

Solve the system numerically assuming the initial conditions: \( y_1(0) = y_2(0) = y_3(0) = 1 \). Take \( k_1 = 1 \) and experiment with values of \( k_2 \) of varying magnitude, specifically \( k_2 = 10, 100 \) and \( 1000 \). For each value of \( k_2 \), solve the system using fourth order Runge-Kutta method and implicit Euler method. Compare the two methods with respect to accuracy and stability for a given step size. In each instance, integrate the ODE system from \( t = 0 \) until the solution approaches a steady state or until the method is clearly unstable or grossly inefficient.

4. (Heath) The following system of ODEs, formulated by Lorentz, represents a crude model of atmospheric circulation:

\[
\begin{align*}
  y_1' &= \sigma(y_2 - y_1) \\
  y_2' &= ry_1 - y_2 - y_2y_3 \\
  y_3' &= y_1y_2 - y_3
\end{align*}
\]

Taking \( \sigma = 10, b = 8/3, r = 28 \) and initial values \( y_1(0) = y_3(0) = 0 \) and \( y_2(0) = 1 \), integrate this ODE from \( t = 0 \) to \( t = 100 \) (use Runge-Kutta fourth order method). Plot each of \( y_1, y_2 \) and \( y_3 \) as a function of \( t \), and also plot each of the trajectories \((y_1(t), y_2(t)), (y_1(t), y_3(t))\) and \((y_2(t), y_3(t))\) as a function of \( t \), each on a separate plot. Try perturbing the initial values by a tiny amount and see how much difference this makes in the final value of \( y(100) \).
5. Consider the following implicit method iteration for solving the ODE \( y' = f(t, y) \), where \( \alpha \) is a parameter between 0 and 1:

\[
y_{k+1} = y_k + h[(1 - \alpha)f(t_k, y_k) + \alpha f(t_{k+1}, y_{k+1})].
\]

Show that the local error is \( O(h^2) \), unless \( \theta = 1/2 \), in which case it is \( O(h^3) \). You can assume that all high order derivatives of \( y \) are bounded, and \( f \) satisfies the following Lipschitz condition for all \( t, y, z \) (in its domain):

\[
|f(t, y) - f(t, z)| \leq L|y - z|.
\]

6. Show that if \( x = x_0 \) is a zero of \( f(x) \) of multiplicity \( m \) (i.e., \( f(x_0) = 0, f'(x_0) = 0, \ldots, f^{(m-1)'}(x_0) = 0, f^{(m)'}(x_0) \neq 0 \)), then the iteration

\[
x_{n+1} = x_n - m \frac{f(x_n)}{f'(x_n)}
\]

converges quadratically under suitable continuity conditions.

7. (Heath) Consider the function \( g(x) = x - x^3 \). Find the unique fixed point of \( g(x) \). Prove that the fixed-point iteration converges to this unique fixed point if the starting point lies in the open interval \((-1, 1)\). Is it true that, for some \( k < 1 \) and all \( n \), \(|e_n| \leq k|e_{n-1}|\)?