COL 726

Homework I

Due on Jan 19, 2018

Use single precision (24 bits) unless specified otherwise.

1. Consider the function $f : \Re^2 \to \Re$ defined by $f(x, y) = x - y$. Measure the size of the input (x, y) by $|x| + |y|$. What is the condition number of this function? When is the condition number very high ? Justify your answer.

2. Consider the expression
$$\frac{1}{1-x} - \frac{1}{1+x},$$
assuming $x \neq \pm 1$.

   (a) For what range of values of x is it difficult to compute this expression in floating point arithmetic ?

   (b) How will you compute accurately the value of this expression in the range of x lying in part (a) ?

3. The *fibonacci* numbers f_k are defined by $f_0 = 1, f_1 = 1$, and
$$f_{k+1} = f_k + f_{k-1} \tag{1}$$
for any integer $k > 1$. A small perturbation of them, the *pib* numbers, p_k, are defined by $p_0 = 1, p_1 = 1$ and
$$p_{k+1} = c \cdot p_k + p_{k-1} \tag{2}$$
for any integer $k > 1$ where $c = 1 + \frac{\sqrt{3}}{100}$.

   (a) Plot the numbers f_n and p_n together in one log scale plot. On the plot, mark $1/\epsilon_{mach}$ for single and double precision arithmetic.

   (b) Rewrite (1) to express f_{k-1} in terms of f_k and f_{k+1}. Use the computed f_n and f_{n-1} to recompute f_k for $k = n-2, n-3, \ldots, 0$. Make a plot of the difference between the original $f_0 = 1$ and the recomputed f_0 as a function of n. What n values result in low accuracy for the recomputed f_0 ? How do the results in single and double precision differ ?

   (c) Repeat part (b) for the pib numbers. Comment on the striking difference in the way precision is lost in the two cases. Explain the results.

4. The polynomial $(x-1)^6$ has the value zero at $x = 1$ and is positive elsewhere. The expanded form of the polynomial, $x^6 - 6x^5 + 15x^4 - 20x^3 + 15x^2 - 6x + 1$, is mathematically equivalent but may not give the same results numerically. Compute and plot the values of this polynomial, using each of the two forms, for 101 equally spaced points in the interval $[0.995, 1.005]$, i.e., with a spacing of 0.0001. Can you explain the difference ?

5. Write a program to generate the first 80 terms in the sequence given by the difference equation $x_{k+1} = 2.25x_k - 0.5x_{k-1}$, with starting values $x_1 = \frac{1}{3}$ and $x_2 = \frac{1}{12}$. Make a semilog plot of the values you obtain as a function of k. The exact solution to the equation above is $x_k = \frac{4^{1-k}}{3}$, which decrease monotonically as k increases. Does your graph confirm this expected behaviour ? Can you explain your results ?

6. Consider the differential equation

$$\begin{aligned} y'(x) &= \frac{2}{\pi} xy(y-\pi), \quad 0 \le x \le 10 \\ y(0) &= y_0 \end{aligned}$$

   (a) Show/verify that the exact solution to this equation is

$$y(x) = \frac{\pi y_0}{y_0 + (\pi - y_0)e^{x^2}}$$

   (b) Taking $y_0 = \pi$ compute the solution for

      i. an 8 digit rounded approximation for π.

      ii. a 9 digit rounded approximation for π.

   What can you say about the results?

7. Consider the problem of determining the value of the integral

$$\int_0^1 x^{20} e^{x-1} dx.$$

   If we let

$$I_k = \int_0^1 x^k e^{x-1} dx,$$

   then integration by part gives

$$\begin{aligned} I_k &= 1 - kI_{k-1} \\ I_0 &= \int_0^1 e^{x-1} dx = 1 - 1/e \end{aligned}$$

   Thus, we can compute I_{20} by successively computing $I_1, I_2, \ldots$. Write a program to compute this, and plot (k, I_k) values. What are the errors due to ?