CSL356 Aug 11,12,14

TUTORIAL SHEET 2

1. You are given a line with n points, labeled 1 to n, marked on it. You are also given a set of intervals $I_1, \ldots, I_k$, where interval I_i is of the form $[s_i, e_i]$, $1 \leq s_i \leq e_i \leq n$. Find a set of points X of smallest cardinality such that each interval contains at least one point from X.

   Solution: Sort the intervals in increasing order of e_i. Select the first point as the right end-point of the first interval in this order. Remove all intervals which intersect with this point, and repeat. Proof of correctness is similar to the interval scheduling problem discussed in class. If the algorithm picks points $p_1 < p_2 < \cdots < p_k$, and optimum solution picks points $q_1 < q_2 < \cdots, q_s$, then prove by induction that $p_i \geq q_i$, and so $k \leq s$.

2. You are given two sets X and Y of n positive integers each. You are asked to arrange the elements in each of the sets X and Y in some order. Let x_i be the i^{th} element of X in this order, and define y_i similarly. Your goal is to arrange them such that $\Pi_{i=1}^{n} x_i^{y_i} = x_1^{y_1} \times x_2^{y_2} \times \cdots \times x_n^{y_n}$ is maximized. Give an efficient algorithm to solve this problem. Prove correctness of your algorithm.

   Solution: Arrange the elements of X in decreasing order of x_i values – let this ordering be $1, 2, \ldots, n$. Do the same for Y, and let the ordering be $1, 2, \ldots, n$. These are the orderings of X and Y produced by the greedy algorithm. You can always assume that the optimum solution orders X as $1, \ldots, n$ as well. Now if it does not order Y as $1, \ldots, n$, then there must be two consecutive elements in the ordering of Y, say i_1, i_2, such that $i_1 > i_2$. Now show that by reversing the ordering, you get a better solution. And then argue as done in class: we reduce the number of inversions with respect to our solution.

3. Suppose you want to go from city A to city B on a long highway. Once you fill your car tank to full capacity, it can travel D kilometres. There are several locations on the highway which have petrol pumps. Assume that there is a petrol pump at the start of the highway, and every stretch of length D on the highway has a location with a petrol pump. Given the location of these petrol pumps, devise a strategy for traveling from A to B so that you will have to make as few stops for filling petrol as possible.

   Solution: Solution is again a greedy algorithm. Let p be the last petrol station at which we filled petrol. Look at the segment of length D with left end-point at p, and choose the next petrol station as the last one in this segment, and so on. Again, the proof is like Question 1 above: if the algorithm fills petrol at $p_1, \ldots, p_k$ (from left to right), and optimum does this at $q_1, \ldots, q_s$, then $p_i \geq q_i$.

Another way of thinking about this problem is to *reduce* it to Problem 1 above: for each petrol station, draw an interval of length D with this petrol station as its right end-point.

4. [KT Chapter 4] Given a list of n natural numbers $d_1, d_2, \ldots, d_n$, show how to decide in polynomial time whether there exists an undirected graph $G = (V, E)$ whose node degrees are precisely the numbers $d_1, \ldots, d_n$. (That is, if $V = \{v_1, \ldots, v_n\}$, then the degree of v_i should be exactly d_i.) G should not contain multiple edges between the same pair of nodes, or "loop" edges with both endpoints equal to the same node.

   Solution: If all the d_i are 0, then we know that there is such a graph: the graph has n vertices and no edges. So assume this is not the case. Sort the d_i in decreasing order: $d_1 \geq d_2 \geq \ldots \geq d_n$. Now argue that there is a graph with degree sequence $(d_1, \ldots, d_n)$ if and only if there is a graph (on $n-1$ vertices) with degree sequence $(d_2 - 1, d_3 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_n)$, where $k = d_1$. In other words, we are saying that if a graph with sequence $(d_1, \ldots, d_n)$ exists, then we can assume that the highest degree vertex (of degree d_1) has edges to the next d_1 highest degree vertices. Let us see why. One direction of the proof is easy: if there is a graph G with degree sequence $(d_2 - 1, d_3 - 1, \ldots, d_k - 1, d_{k+1}, \ldots, d_n)$, then there is a graph with degree sequence $(d_1, \ldots, d_n)$: add a new vertex to G which has edges to the vertices with degrees $d_2 - 1, d_3 - 1, \ldots, d_k - 1$. Let us now prove the reverse (and the more non-trivial direction of the proof). Suppose there is a graph G with degree sequence $(d_1, \ldots, d_n)$. Let v_i be the vertex with degree d_i. If v_1 has edges to $v_2, \ldots, v_k$ in G, then we are done – just remove v_1 and you have the graph with the desired degree sequence. So assume there is an index i, $2 \leq i \leq k$ such that (v_1, v_i) is not an edge. Since degree of v_1 is k ($=d_1$), there must be an index $j > k$ such that (v_1, v_j) is an edge. Since degree of v_i is at least that of v_j, there must be a vertex v_k such that (v_i, v_k) is an edge, but (v_j, v_k) is not an edge. Now, in G, we remove the edges (v_1, v_j) and (v_i, v_k), and add the edges (v_1, v_i) and (v_j, v_k). Note that this does not change the degree of any vertex, but now, we have increased the number of edges from v_1 to the vertices in the set $\{v_2, \ldots, v_k\}$. Repeat this process till v_1 has edges to $\{v_2, \ldots, v_k\}$.