CSL361 Problem set 8: Householder and Givens Orthogonalization

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1 Householder QR

Let $v \in \mathbb{R}^n$ be non-zero. The $n \times n$ matrix (rank 1 corrections to the identity)

$$P = I - 2\frac{vv^T}{v^T v}$$

is called the *Householder reflection*. Show that:

1. $P = P^t = P^{-1}$.

2. When a vector $x \in \mathbb{R}^n$ is multiplied by P, it is reflected across the hyperplane $span\{v\}^\perp$.

3. If
$$v = x \pm \|x\|_2 e_1$$
Then,
$$Px = \mp \|x\|_2 e_1 \in span\{e_1\}$$
That is, a *House holder reflection* along $v = x \pm \|x\|_2 e_1$ annihilates all but the first component of x while preserving its norm.

4. Argue that out of the two choices for v for annihilating all but the first component of x, $v = x + sign(x_1)\|x\|_2 e_1$ is better from a numerical point of view. Explain the two choices geometrically.

5. If $P = I - 2vv^T/v^T v$ and A is a matrix, then show that
$$PA = \left(I - 2\frac{vv^T}{v^T v}\right) A = A + vw^t$$

where $w = \beta A^t v$ and $\beta = -2/v^t v$. Thus, a Householder update of a matrix involves a matrix-vector multiplication followed by a outer-product update. Argue that treating P as a general matrix increases work by an order of magnitude.

Also show that the above produces a Householder vector $\hat{v}$ near to the exact v. That is if $\hat{P} = I - 2\hat{v}\hat{v}^T/\hat{v}^T\hat{v}$, then

(a) $\|\hat{P} - P\|_2 = O(\mu)$

(b) $fl(\hat{P}A) = P(A + E)$, where $\|E\|_2 = O(\mu\|A\|_2)$

6. More generally, for a given m-vector x, consider the partioning

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

where x_1 is a $(k-1)$-vector, $1 \leq k < m$. Show that if we take the Householder vector to be

$$v = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} - \alpha e_k$$

where $\alpha = -sign(x_k)\|x_2\|_2$ (x_k being the k^{th} component of x), then the resulting Householder transformation annihilates the last $m - k$ components of x.

7. Using the above results show that a sequence of Householder transformations can be defined for $k = 1, \ldots, n$ such that all the subdiagonal entries of matrix are annihilated. Show that such a sequence defines a QR factorization.

2 Givens rotations

Householder reflections are useful for intrducing zeros on a grand scale. However, in situations where it is necessary to zero elements more selectively, Givens rotations are useful. These are rank 2 corrections to identity of the form

$$G(i,k,\theta) = \begin{bmatrix} 1 & \dots & 0 & \dots & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & & \vdots & & \vdots \\ 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots & & \vdots \\ 0 & \dots & -s & \dots & c & \dots & 0 \\ \vdots & & \vdots & & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \dots & 0 & \dots & 1 \end{bmatrix}$$

where $c = \cos\theta$ and $s = \sin\theta$ for some θ.

1. Show that Givens rotations are orthogonal. In particular, show that premultiplication by $G(i,k,\theta)^t$ amounts to a counterclockwise rotation by θ radians in the (i,k) coordinate plane.

2. Show that k^{th} component of a vector can be set to 0 by premultiplication with $G(i,k,\theta)^t$ where

$$c = \frac{x_i}{\sqrt{x_i^2 + x_k^2}} \qquad s = \frac{-x_k}{\sqrt{x_i^2 + x_k^2}}$$

   Show also that only rwos i and k are affected.

3. Describe a procedure for Given's QR.

4. Show that the numerical properties of Givens QR are as favourable as Householder QR.