CSL361 Problem set 7: Error analysis of LU and some extensions

April 6, 2015

1. Assume that A is an $n \times n$ matrix of floating point numbers. If no zero pivot is encountered during the execution of LU decomposition (without pivoting), then show that the computed triangular matrices $\hat{L}$ and $\hat{U}$ satisfy

$$\hat{L}\hat{U} = A + H$$
$$|H| \leq 3(n-1)\mu(|A| + |\hat{L}||\hat{U}|) + O(\mu^2)$$

2. Suppose the $\hat{L}$ and $\hat{U}$ computed as above are used in backward and forward substitutions to obtain computed solutions $\hat{y}$ and $\hat{x}$ to $\hat{L}y = b$ and $\hat{U}x = \hat{y}$ respectively. Show that then $(A+E)\hat{x} = b$ with

$$|E| \leq n\mu(3|A| + 5|\hat{L}||\hat{U}|) + +O(\mu^2)$$

3. *Gauss-Jordan* elimination is a variation of standard Gaussian elimination in which the matrix is reduced to a diagonal form than merely to a traingular form. The elimination matrix used for a given column vector a is of the form

$$\begin{bmatrix} 1 & \dots & 0 & -m_1 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 1 & -m_{k-1} & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & -m_{k+1} & 1 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & -m_n & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_{k-1} \\ a_k \\ a_{k+1} \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ a_k \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where $m_i = a_1/a_k, i = 1 : n$. Under what situations will *Gauss-Jordan* elimination be useful? What is its work-load? What are the disadvantages? What can you say about its numerical stability?

4. Show that if all the leading principal submatrices of $A \in \mathbb{R}^{n \times n}$ are nonsingular, then there exists unique lower-triangular matrices L and M and a unique diagonal matrix D such that $A = LDM^t$. How can the factorization be computed?

5. Show that if $A = LDM^t$ and A is symmetric, then $L = M$.

6. A matrix A is *positive definite* if $x^t Ax > 0$ for all non-zero $x \in \mathbb{R}^n$. Show that if A is positive definite then it is non-singular.

7. Show that if $A \in \mathbb{R}^{n \times n}$ is positive definite and $X \in \mathbb{R}^{n \times k}$ has rank k, then $B = X^t AX \in \mathbb{R}^{k \times k}$ is also positive definite.

8. Show that if A is positive definite then all its principal matrices are also positive definite. In particular, show that all diagonal entries are positive.

9. Show that for a positive definite A the factorization $A = LDM^t$ exists and all elements of D are positive.

10. **Cholesky factorization:** Show that if $A \in \mathbb{R}^{n \times n}$ is symmetric and positive definite, then there exists a unique lower-triangular matrix $L \in \mathbb{R}^{n \times n}$ with positive diagonal entries such that $A = LL^t$. Give an algorithm for Cholesky factorization.

11. Show that if A is a symmetric matrix and P is a permutation matrix, then the update PAP^t preserves symmetry. Use the above to give an algorithm for Cholesky factorization with pivoting.

12. Given $A \in \mathbb{R}^{m \times n}$ with the property that $rank(A) = n$ and $b \in \mathbb{R}^m$, show that the following algorithm computes the least square solution $min||Ax - b||_2$.

    Compute the lower triangular portion of $C = A^t A$
    $d = A^t b$
    Compute the Cholesky factorization$C = LL^t$
    Solve $Ly = d$ and $L^t x = y$

    Show that the least square solution is unique. What is the total workload? Is there any advantage over standard Gaussian elimination?

13. Let $A \in \mathbb{R}^{m \times n}$ with $m \geq n$ and $b \in \mathbb{R}^m$ be given and suppose that an orthogonal matrix $Q \in \mathbb{R}^{m \times m}$ has been computed such that

$$Q^t A = R = \begin{bmatrix} R_1 \\ 0 \end{bmatrix} \begin{matrix} n \\ m-n \end{matrix}$$

is upper-triangular. If

$$Q^t b = \begin{bmatrix} c \\ d \end{bmatrix} \begin{matrix} n \\ m-n \end{matrix}$$

show that,

$$||Ax - b||_2^2 = ||Q^t Ax - Q^t b||_2^2 = ||R_1 x - c||_2^2 + ||d||_2^2$$

for any $x \in \mathbb{R}^n$. Also, if $rank(A) = rank(R_1) = n$ then the least square solution is defined by $R_1 x = c$ with a residual error of $||d||_2^2$. What is the total-work-load if the QR factorization is computed using *Modified Gram-Schmidt*? How does it compare with Cholesky?