CSL361 Problem set 4: Basic linear algebra

February 21, 2017

[**Note:**] If the numerical matrix computations turn out to be tedious, you may use the function **rref** in **Matlab**.

1 Row-reduced echelon matrices

1. Consider the following systems of equation

   (a)
$$\begin{array}{rcrcrcl} x_1 & - & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & & & + & 2x_3 & = & 1 \\ x_1 & - & 3x_2 & + & 4x_3 & = & 2 \end{array}$$

   (b)
$$\begin{array}{rcrcrcrcl} x_1 & - & 2x_2 & + & x_3 & + & 2x_4 & = & 1 \\ x_1 & + & x_2 & - & x_3 & + & x_4 & = & 2 \\ x_1 & + & 7x_2 & - & 5x_3 & - & x_4 & = & 3 \end{array}$$

   Find out whether they have solutions. If so, describe explicitly all solutions.

2. Let
$$\mathbf{A} = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}.$$
   for which triples (y_1, y_2, y_3) does the system $\mathbf{AX} = \mathbf{Y}$ have a solution?

3. Let
$$\mathbf{A} = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$
   for which (y_1, y_2, y_3, y_4) does the system $\mathbf{AX} = \mathbf{Y}$ have a solution?

4. Suppose $\mathbf{R}$ and $\mathbf{R}'$ are 2×3 row-reduced echelon matrices and that the systems $\mathbf{RX} = \mathbf{0}$ and $\mathbf{R}'\mathbf{X} = \mathbf{0}$ have exactly the same solutions. Prove that $\mathbf{R} = \mathbf{R}'$

5. Let

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -1 & 0 & 3 & 5 \\ 1 & -2 & 1 & 1 \end{bmatrix}$$

Find a row-reduced echelon matrix $\mathbf{R}$ which is row-equivalent to $\mathbf{A}$ and an invertible 3×3 matrix $\mathbf{P}$ such that $\mathbf{R} = \mathbf{PA}$.

6. For each of the three matrices

$$\begin{bmatrix} 2 & 5 & -1 \\ 4 & -1 & 2 \\ 6 & 4 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

use elementary row operations to discover whether it is invertible, and to find the inverse in case it is.

7. Suppose $\mathbf{A}$ is a 2×1 matrix and that $\mathbf{B}$ is a 1×2 matrix. Prove that $\mathbf{C} = \mathbf{AB}$ is not invertible.

8. Let $\mathbf{A}$ be an $n \times n$ matrix. Prove the following:

   (a) If $\mathbf{A}$ is invertible and $\mathbf{AB} = \mathbf{0}$ for some $n \times n$ matrix $\mathbf{B}$, the $\mathbf{B} = \mathbf{0}$.

   (b) If $\mathbf{A}$ is not invertible, then there exists an $n \times n$ matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{0}$ but $\mathbf{B} \neq \mathbf{0}$.

9. Let

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Prove, using elementary row operations, that $\mathbf{A}$ is invertible *if and only if* $(ad - bc) \neq 0$.

10. Let

$$\mathbf{A} = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$$

For which $\mathbf{X}$ does there exist a scalar c such that $\mathbf{AX} = c\mathbf{X}$?

11. An $n \times n$ matrix $\mathbf{A}$ is **upper-triangular** if $A_{ij} = 0$ for $i > j$. Prove that $\mathbf{A}$ is invertible *if and only if* every entry on its main diagonal is distinct from 0.

12. Prove that if $\mathbf{A}$ is an $m \times n$ matrix, $\mathbf{B}$ is an $n \times m$ matrix and $n < m$, then $\mathbf{AB}$ is not invertible (generalization of a previous problem).

13. Let $\mathbf{A}$ be an $m \times n$ matrix. Show that by means of a finite number of elementary row and/or column operations one can pass from $\mathbf{A}$ to a matrix $\mathbf{R}$ which is both *row-reduced echelon* and *column-reduced echelon*, i.e., $R_{ij} = 0$ if $i \neq j$, $R_{ii} = 1, 1 \leq i \leq r$, $R_{ii} = 0, i > r$. Show that $\mathbf{R} = \mathbf{PAQ}$ where $\mathbf{P}$ is an invertible $n \times n$ matrix and $\mathbf{Q}$ is an invertible $m \times m$ matrix.

2 Vector spaces and subspaces

1. Show that the following are vector spaces:

   (a) The n-tuple space, F^n: Let F be a Field and let V be the set of all n-tuples $\alpha = (x_1, x_2, \ldots, x_n)$ of scalars $x_i \in F$. If $\beta = (y_1, y_2, \ldots, y_n)$ with $y_i \in F$ then their sum is defined as

   $$\alpha + \beta = (x_1 + y_1, \ldots, x_n + y_n)$$

   and the product of a scalar c and a vector α is

   $$c\alpha = (cx_1, cx_2, \ldots, cx_n)$$

   (b) The space of $m \times n$ matrices, $F^{m \times n}$: under usual matrix addition and multiplication of a matrix with a scalar.

   (c) The space of functions from a set to a Field: under the operations:

   $$(f + g)(s) = f(s) + g(s)$$

   and

   $$(cf)(s) = cf(s)$$

   (d) The space of polynomial functions over a Field: with addition and scalar multiplication as defined above.

2. Which of the following sets of vectors $\alpha = (a_1, \ldots, a_n)$ in $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$ $(n \geq 3)$?

(a) all α such that $a_1 \geq 0$;

(b) all α such that $a_1 + 3a_2 = a_3$;

(c) all α such that $a_2 = a_1^2$;

(d) all α such that $a_1 a_2 = 0$;

(e) all α such that a_2 is rational.

3. Let V be the (real) vector space of all functions $f : \mathbb{R} \to \mathbb{R}$. Which of the following are subspaces of V?

   (a) all f such that $f(x^2) = f(x)^2$;

   (b) all f such that $f(0) = f(1)$;

   (c) all f such that $f(3) = 1 + f(-5)$;

   (d) all f such that $f(-1) = 0$;

   (e) all f which are continuous.

4. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ which satisfy

$$\begin{array}{rcrcrcrcrcl} 2x_1 & - & x_2 & + & \frac{4}{3}x_3 & - & x_4 & & & = & 0 \\ x_1 & & & + & \frac{2}{3}x_3 & & & - & x_5 & = & 0 \\ 9x_1 & - & 3x_2 & + & 6x_3 & - & 3x_4 & - & 3x_5 & = & 0 \end{array}$$

   Find a finite set of vectors which spans W.

5. Let F be a Field and let n be a positive integer ($n \geq 2$). Let V be a vector space of all $n \times n$ matrices over F. Which of the following set of matrices A in V are subspaces of V?

   (a) all invertible A;

   (b) all non-invertible A;

   (c) all A such that $AB = BA$, where B is some fixed matrix in V;

   (d) all A such that $A^2 = A$.

6. (a) Prove that the only subspaces of $\mathbb{R}^1$ are $\mathbb{R}^1$ and the zero subspace.

   (b) Prove that a subspace of $\mathbb{R}^2$ is $\mathbb{R}^2$, or the zero subspace, or consists of all scalar multiples of some fixed vector in $\mathbb{R}^2$.

   (c) What can you say about the subspaces of $\mathbb{R}^3$?

7. Let $\mathbf{A}$ be a $m \times n$ matrix over F. Show that the set of all vectors $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{0}$ is a subspace of F^n. This subspace is called the **null space** of $\mathbf{A}$ and its dimension in the **nullity** of $\mathbf{A}$.

8. Let **A** be a $m \times n$ matrix over F. Show that the set of all vectors spanned by the row vectors of **A** is a subspace of F^n. This subspace is called the **row space** of **A** and its dimension in the **row rank** of **A**.

9. Let **A** be a $m \times n$ matrix over F. Show that the set of all vectors **Y** such that $\mathbf{AX} = \mathbf{Y}$ has a solution for **X** is a subspace of F^m. This subspace is called the **range space** of **A** and its dimension in the **column rank** of **A** (why?).

10. Show that for any matrix **A**

    (a) **nullity** + **row rank** $= n$

    (b) **nullity** + **column rank** $= n$

    and conclude that
    row rank of **A** = **column rank** of **A**

11. Consider the 5×5 matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

    (a) Find an invertible matrix **P** such that **PA** is a row-reduced echelon matrix **R**.

    (b) Find a basis for the **row space** W of **R**.

    (c) Say which vectors $(b_1, b_2, b_3, b_4, b_5)$ are in W.

    (d) Find the coordinate matrix of each vector $(b_1, b_2, b_3, b_4, b_5) \in W$ in the ordered basis chosen in (b).

    (e) Write each vector $(b_1, b_2, b_3, b_4, b_5) \in W$ as a linear combination of the rows of **A**.

    (f) Give an explicit description of the **null space** of **A**.

    (g) Find a basis for the **null space**.

    (h) For what column matrices **Y** does the equation $\mathbf{AX} = \mathbf{Y}$ have solutions **X**?

    (i) Explicitly find the **range space** of **A** and find a basis.

    (j) Verify the relations regarding the **nullity**, **row rank** and **column rank** of **A**.

3 Coordinates

1. Let $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$ be an ordered basis for $\mathbb{R}^3$ where

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 1, 1), \quad \alpha_3 = (1, 0, 0)$$

   What are the coordinates of (a, b, c) in the ordered basis $\mathcal{B}$?

2. Let $\alpha = (x_1, x_2)$ and $\beta = (y_1, y_2)$ be vectors in $\mathbb{R}^2$ such that

$$x_1 y_1 + x_2 y_2 = 0 \text{ and } x_1^2 + x_2^2 = y_1^2 + y^2 = 1$$

   Show that $\mathcal{B} = \{\alpha, \beta\}$ is a basis for $\mathbb{R}^2$. Find the coordinates of (a, b) in this ordered basis. What do the conditions mean geometrically?

3. Consider the matrix

$$\mathbf{P} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

   Show that $\mathbf{P}$ is invertible. Conclude that $\mathbf{P}$ represents a transformation of coordinates in $\mathbb{R}^2$. What is the geometric interpretation of the change of coordinates represented by $\mathbf{P}$?

4. Let V be a vector space over the complex numbers of all functions from $\mathbb{R}$ to $\mathtt{C}$, i.e., the space of all complex valued functions on the real line. Let $f_1(x) = 1$, $f_2(x) = e^{ix}$ and $f_3(x) = e^{-ix}$.
   (a) Prove that f_1, f_2, f_3 are linearly independent.
   (b) Let $g_1(x) = 1$, $g_2(x) = \cos x$ and $g_3(x) = \sin x$. Find an invertible 3×3 matrix $\mathbf{P}$ such that

$$g_j = \sum_{i=1}^{3} P_{ij} f_i$$

5. Let V be the real vector space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less. Let t be a fixed number and define

$$g_1 x = 1, \quad g_2(x) = x + t, \quad g_3(x) - (x + t)^2$$

   Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

   what are the coordinates of f in this ordered basis $\mathcal{B}$?