CS130N Problem set 2: Asymptotic orders of growth

January 22, 2001

1. Let $f(n)$ and $g(n)$ be asymptotically non-negative functions. Show that $max(f(n), g(n)) = \Theta(f(n) + g(n))$.

2. Show that for any real constants a and b, $b > 0$,
$$(n + a)^b = \Theta(n^b)$$

3. Explain why the statement, "The running time of insertion sort is at least $O(n^2)$," is content-free.

4. What do we mean in each of the following?
   (a) $2n^2 + 3n + 1 = 2n^2 + \Theta(n)$
   (b) $T(n) = 2T(n/2) + \Theta(n)$ (why is it acceptable not to write the base case for this recurrence?)
   (c) $\sum_{i=1}^{n} O(i)$
   (d) $2n^2 + \Theta(n) = \Theta(n^2)$

5. Consider the recurrence $T(1) = 1$, $T(n) = 2T(n/2)+n$. What is wrong in the following proof by induction that $T(n) = O(n)$?

   We prove $T(n) \leq cn$. We obviously have the base case. Now,
$$\begin{aligned} T(n) &\leq 2(cn/2) + n \quad \text{by I.H.} \\ &= cn + n \\ &= O(n) \end{aligned}$$

6. Is $2^{n+1} = O(2^n)$? Is $2^{2n} = O(2^n)$?

7. Show that $O(2^{\log_a(n)})$ is **not** the same as $O(2^{\log_b(n)})$, unless $a = b$ of course.

8. Show that $\log(n!) = \Theta(n \log n)$.

9. For which of the following functions is it true that $f(2n) = O(f(n))$?

   (a) $f(n) = n$
   (b) $f(n) = n^4$
   (c) $f(n) = 2^n$
   (d) $f(n) = \log n$
   (e) $f(n) = n \log n$

10. In a certain population of amoebae, there are "active" amoebae and "passive" amoebae. Every minute, each active amoebae is turned into 3 active amoebae and 7 new passive ones. All the old passive amoebae remain unchanged. Starting with a single active amoeba, how many amoebae will there be after n minutes? Show that this is $\Theta(3^n)$.

11. Use the fact that $\dfrac{d}{dx} \log\log x = \dfrac{1}{x \log x}$ to show that

$$\sum_{k=2}^{n} \frac{1}{k \log k} = O\left(\log\log n\right).$$

12. Prove or disprove:

   (a) $f(n) = \Theta(g(n))$ and $g(n) = \Theta(h(n))$ implies $f(n) = \Theta(h(n))$.
   (b) $f(n) = O(g(n))$ and $g(n) = O(h(n))$ implies $f(n) = O(h(n))$.
   (c) $f(n) = \Omega(g(n))$ and $g(n) = \Omega(h(n))$ implies $f(n) = \Omega(h(n))$.
   (d) $f(n) = O(g(n))$ implies $g(n) = O(f(n))$.
   (e) $f(n) = O(g(n))$ iff $g(n) = \Omega(f(n))$.
   (f) $f(n) = O(g(n))$ implies $g(n) = \Omega(f(n))$.
   (g) $f(n) = O(g(n))$ implies $2^{f(n)} = O(2^{g(n)})$.
   (h) $f(n) = \Theta(g(n))$ iff $g(n) = \Theta(f(n))$

13. Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on nonnegative integers by the recurrence

$$T(n) = aT(n/b) + f(n)$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Show that $T(n)$ can be bounded asymptotically as follows:

(a) If $f(n) = O(n^{log_b a - \epsilon})$ for some $\epsilon > 0$, then $T(n) = \Theta(n^{log_b a})$.

(b) If $f(n) = \Theta(n^{log_b a})$, then $T(n) = \Theta(n^{log_b a} \lg n)$

(c) If $f(n) = \Omega(n^{log_b a + \epsilon})$ for some $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some $c < 1$ and all sufficiently large n, the $T(n) = \Theta(f(n))$.

[Note: The above is called the **master theorem**. The tutors may discuss some applications.]