CSL 356 Aug 27 Lec 14

S: ground set with elements $e_1, e_2, \ldots e_n$

M: A family of subsets that are "independent" or "feasible"

$\phi \in M$ $\quad M \subseteq 2^S$ (powerset)

W: weight function $S \to R^+$

Objective: Find a subset $S \in M$ such that $W(S)$ is maximum

$W(S)$ = sum of weights of elements in S

Algorithm Basic Greedy

Consider elements of S in decreasing order of weights, say $\tilde{e}_n, \tilde{e}_{n-1} \ldots \tilde{e}_1$

$T = \phi$

For $i := n$ down to 1 do

If $T \cup \{e_i\}$ is "independent"

$T \leftarrow T \cup \{e_i\}$

Output T

Under what conditions Basic Greedy provides an optimal solution $\longrightarrow$ for any instance of the problem and under any weight function

Subset Property of (S, M) implies that if $S_1 \in M$ then for any $S_2 \subset S_1$, $S_2 \in M$

<u>Claim</u> The following are equivalent

1. Basic greedy solves the problem for (S, M). In this case (S, M) is called a "<u>Matroid</u>".

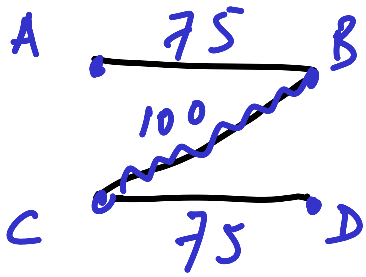
2. For any subsets $S_i, S_{i+1} \in M$ where $|S_i| = i$ $|S_{i+1}| = i+1$, there exists an $e \in S_{i-1} - S_i$, s.t. $S_i \cup \{e\} \in M$
(exchange property)

3. For any $A \subset S$, the size of maximal* subsets of A are identical.

* maximal : no element can be added from A and maintain feasibility.

$1 \leftrightarrows 2 \qquad 1 \leftrightarrows 3 \qquad 2 \leftrightarrows 3$

$1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$

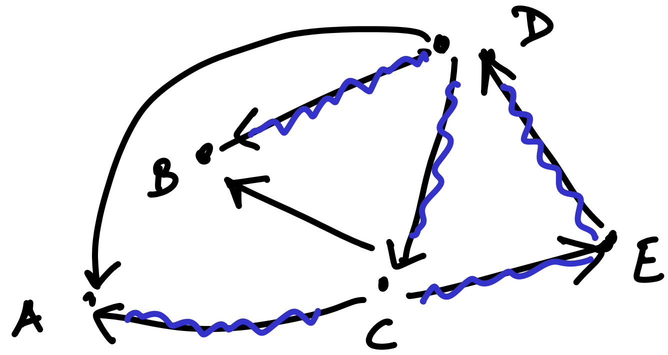


$S = \{AB, BC, CD\}$

$M : \{AB\}, \{CD\}, \{BC\}$

$\{AB, CD\}, \emptyset$

$S_1 = \{BC\} \qquad S_2 = \{AB, CD\}$



It satisfies exchange property

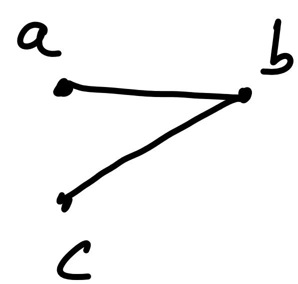
Maximal Spanning Tree

S_1 : is a forest with i edges

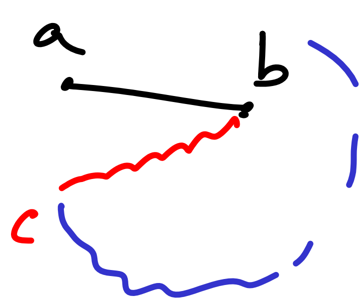
S_2 : is a forest with $i+1$ edges

Claim: $\exists\, e \in S_2 - S_1$ s.t. $S_1 \cup \{e\}$ is a forest

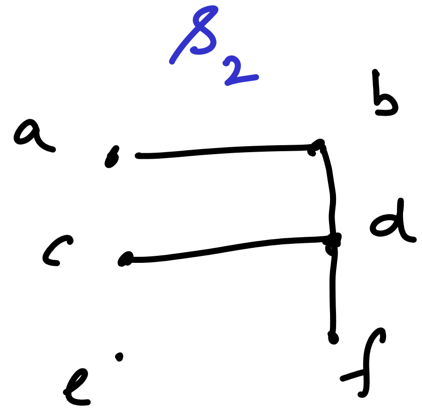
S_2

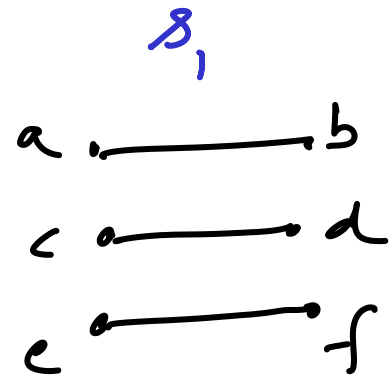


S_1



Case 1 $\exists$ a vertex (induced by the edges in S_2) that doesn't belong to S_1. Any edge incident on that vertex cannot create cycle in S_1

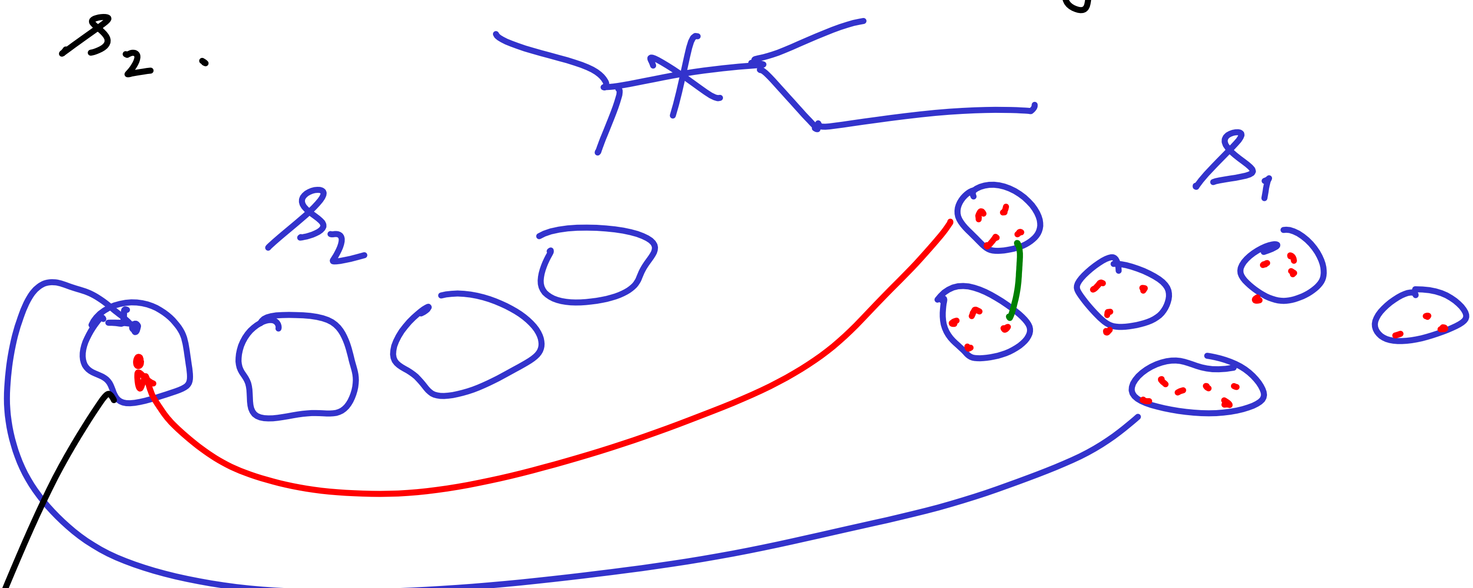




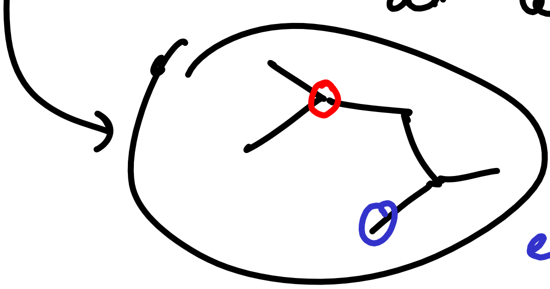
$|S_2| = 4$ $\qquad\qquad$ $|S_1| = 3$

$$V(S_2) \subset V(S_1)$$

The number of connected components (trees) in S_1 is strictly more than S_2.



<u>Observation</u>: Some component of S_2 will have at least 2 colours



We can add an edge from S_2 to S_1 with endpoints in different components