CSL705/CS355N: Theory of Computation

Tutorial sheet: Strings, Numbers and Encodings

1. The decimal representation of natural numbers uses the alphabet $D_0 = \{0, 1, \ldots, 9\}$ to represent the natural numbers as elements of D_0^*. Each digit symbol also has an appropriate *face value* and a *place value* in the string. An alternative representation uses instead the alphabet $D_T = (D_0 - \{0\}) \cup \{T\}$ where the symbol T has a place value of 10 and the other digits have the same place value as usual.
   (a) Prove that there is a bijection between D_T^* and $\mathbb{N}$. Hence every string represents a unique natural number and every natural number has a unique representation in D_T^*.
   (b) Design an algorithm to convert from the representation D_0^* to D_T^*.
   (c) Design algorithms (without converting between representations D_T^* and D_0^*) for addition, subtraction, multiplication, quotient and remainder operations for positive integers represented in D_T^*.

2. Let D be a set. An injective function $\alpha : D \to \mathbb{N}$ is said to be a **natural encoding** of D in $\mathbb{N}$ and for each $d \in D$, $\alpha(d)$ is called a **natural representation** of d. The inverse function α^{-1} is called a **natural decoding**. Let $f : D \to D$ be a total function. Then $f^* = \alpha \circ f \circ \alpha^{-1}$ is called a **natural coding** of f.
   (a) Define $z2n : \mathbb{Z} \to \mathbb{N}$ that codes the set $\mathbb{Z}$ of all integers.
   (b) Define functions that are natural codings of the operations of addition, subtraction and multiplication on the integers.
   (c) Define the quotient and remainder functions on integers.

3. Let Σ be a finite alphabet. Find a bijection between Σ^* and the set $\mathbb{N}$ of natural numbers and prove that it is indeed a bijection.

4. Let $\mathcal{G}(\Lambda)$ be the set of directed finite graphs with labels drawn from a nonempty finite set Λ.
   (a) Define a string representation of the graphs in $\mathcal{G}(\Lambda)$.
   (b) What are the representations of various trivial graphs such as the empty graph, a graph with a single node, a graph with a single node and a self-loop etc.
   (c) Answer the following questions about your representation and prove your answers.
      i. Is your representation injective? If not then your representation is not an encoding and you need to try a different representation.
      ii. Is your representation bijective?
      iii. If your representation is not surjective, what can you say about the function f^* for any given total function f on $\mathcal{G}(\Lambda)$.
   (d) Without converting between graphs and their representations, define the operations of
      i. adding a node to a graph G
      ii. adding an edge to G
      iii. deleting a node from G
      iv. deleting an edge from G