CSL705/CS355N: Theory of Computation

Tutorial: Context-Free Grammars

1. Prove that the following languages over the alphabet Σ are context-free.
   (a) $L_1 = \{a^m b^n \mid m \neq n\}$
   (b) $L_2 = \{a^m b^n \mid m < n\}$
   (c) $L_3 = \{a^m b^n \mid m \geq n\}$
   (d) $L_4 = \{a^m b^n \mid n = 2m\}$
   (e) $L_5 = \{a^m b^n \mid n \neq 2m\}$

2. Consider the context-free grammar $G = \langle V, \Sigma, S, P\rangle$ where $V = \{S\}$, $\Sigma = \{a, b\}$ and P is defined by the rules
   $$S \to \varepsilon \mid aSbS \mid bSaS$$
   (a) Define the language generated by G.
   (b) The grammar is ambiguous. Display two different derivation trees for the same word generated by the grammar.
   (c) *Without adding any new terminal symbols*, define an unambiguous grammar $G' = \langle V', \Sigma, S, P'\rangle$ that generates the same language.
   (d) Prove that $\mathcal{L}(G) = \mathcal{L}(G')$

3. Given a nonempty finite alphabet Σ.
   (a) Design an unambiguous context-free grammar G_1 to generate the language $L = \{ww^R \mid w \in \Sigma^*\}$.
   (b) Prove that the grammar G_1 is unambiguous.
   (c) Let the set of *palindromes* over Σ be defined as $M = \{w \in \Sigma^* \mid w = w^R\}$. What is the set $M - L$?
   (d) Design an unambiguous context-free grammar G_2 to generate M.
   (e) Prove that the grammar G_2 is unambiguous.

4. Consider the grammar $G = \langle\{S\}, \Gamma, S, P\rangle$ where $\Gamma = \{\mathtt{a}, \mathtt{i}, \mathtt{t}, \mathtt{e}\}$ and the productions are
   $$S \longrightarrow \mathtt{a} \mid \mathtt{i}S\mathtt{t}S \mid \mathtt{i}S\mathtt{t}S\mathtt{e}S$$
   (a) Give an equivalent grammar G' in Chomsky Normal Form.
   (b) Prove that the grammar G is ambiguous.
   (c) Use a single pair of brackets $\Pi = \{[,]\}$ to remove this ambiguity by defining a modified grammar $G'' = \langle V'', \Gamma \cup \Pi, S, P''\rangle$ such that $e(\mathcal{L}(G'')) = \mathcal{L}(G)$ where $e : (\Gamma \cup \Pi)^* \longrightarrow \Gamma^*$ is the erasure homomorphism defined by $e([) = e(]) = \varepsilon$ and $h(c) = c$ for each $c \in \Gamma$. *(Note: You don't need to prove that $e(\mathcal{L}(G'')) = \mathcal{L}(G)$, it should be obvious!).*
   (d) Prove that G'' is unambiguous.