CSL105: Discrete Mathematical Structures
I semester 2008-09
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Tutorial sheet: **Sets, Relations and Partitions**

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1. Let A, B and C be sets. Prove that
   (a) $A \cap \bar{B} = A \cap \bar{C}$ if and only if $A \cap B = A \cap C$.
   (b) If $A \cup B \subseteq A \cup C$ and $A \cap B \subseteq A \cap C$ then $B \subseteq C$.
   (c) Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$. The does it necessarily imply that $B = C$? Prove your answer or give a counterexample to show that that $B = C$ does not necessarily hold.

2. Let the **symmetric difference** of two sets A and B be defined as $A \triangle B = (A - B) \cup (B - A)$. What is the relationship between the following pairs of sets given that X, Y and Z are themselves sets? Prove your answers.
   (a) $X \triangle (Y \cup Z)$ and $(X \triangle Y) \cup (X \triangle Z)$.
   (b) $X \triangle (Y \cap Z)$ and $(X \triangle Y) \cap (X \triangle Z)$.

3. Prove that the relation $\mathsf{divisorOf} = \{(a, b)|\ a \text{ is a divisor of } b\}$ is a partial order on the set $\mathbb{P}$ of positive integers.

4. Consider the following relation R on the set $\mathbb{C}$ of complex numbers. $R = \{(w, z) \in \mathbb{C} \times \mathbb{C}|\ |w| \leq |z|\}$. Determine whether this relation is a preorder. Is it a partial order? If so prove it, if not give an example to show that it is not partial order.

5. For any relation $R \subseteq A \times B$, the *converse* of R (denoted R^{-1}) is defined as the relation $\{(b, a)|(a, b) \in R\}$.
   (a) Prove that a relation on a set is symmetric if and only if it equals its converse.
   (b) Prove that the converse of a preorder is a preorder and the converse of a partial order is a partial order.

6. A relation R on a set A is called a *total order* if R is a partial order on A such that $(a, b) \in R$ or $(b, a) \in R$ for each $a, b \in A$. In general a binary relation is *total* if for any two elements $a, b \in A$, $a = b$ or $(a, b) \in R$ or $(b, a) \in R$. A set A is said to be *linearly ordered* by a transitive relation R on A, if for every $a, b \in A$, exactly one of the following conditions holds:

   $$a = b \qquad\qquad (a, b) \in R \qquad\qquad (b, a) \in R$$

   (a) Prove that any linear order on A is irreflexive
   (b) Prove that $L = R - Id_A$ is a linear order if R is a total order.
   (c) Prove that a relation is a linear order iff it is total, irreflexive and transitive.

7. A binary relation on a set is said to be *compatible* if it is reflexive and symmetric. Let R and S be compatible relations on a set A. Which of the following are compatible relations? In each case, either prove that the relation is compatible or construct an example and show that it is not compatible.
   (a) R^{-1} the converse of R (see 5).
   (b) $R \cup S$.
   (c) $R \cap S$.

8. Let $\precsim\ \subseteq A \times A$ be a preorder on A. The set $\cong = \{(a, b) \mid a \precsim b \text{ and } b \precsim a\}$ is called the *kernel* of the preorder $\precsim$.
   (a) Prove that the kernel $\cong$ of the preorder $\precsim$ on A is an equivalence relation on A.

(b) Let $\sqsubseteq \subseteq A/\cong \times A/\cong$ be the relation on the set of equivalence classes of A defined by

$$\sqsubseteq = \{([a]_{\cong}, [b]_{\cong}) \mid a \precsim b\}$$

Prove that $\sqsubseteq$ is a partial order on $A/\cong$.

(c) What is the kernel of each of the following preorders?

i. The $\leq$ relation on the set $\mathbb{R}$ of real numbers.

ii. The $\mathsf{divisorOf}$ relation on the set $\mathbb{P}$.

iii. The relation R in problem 4.

9. Let $R \subseteq A \times B$ be a relation. Define for any $A' \subseteq A$, the set $R(A') = \{b \in B | \exists a \in A' : (a, b) \in R\}$.

(a) Prove that for all $A_1, A_2 \subseteq A$,

Monotonicity. $A_1 \subseteq A_2$ implies $R(A_1) \subseteq R(A_2)$.

Union preservation $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$

Intersection preservation $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

(b) What can you say about the relationship between $R(A-A_1)$ and $R(A-A_2)$ when $A_1, A_2 \subseteq A$?

10. Let $\sqsubseteq$ denote the refinement relation on partitions. Let $\mathbb{Z}$ be the set of integers and for each $k \in \mathbb{P}$, let $\mathbb{Z}/=_k$ denote the set of equivalence classes of the integers modulo k. Prove that for all $k, m \in \mathbb{P}$, $\mathbb{Z}/=_k \sqsubseteq \mathbb{Z}/=_m$ if and only if k is a multiple of m.