CSL105: Discrete Mathematical Structures
I semester 2008-09
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Tutorial sheet: **Graph Theory**

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1. Prove that any undirected graph with $n > 0$ vertices and $0 < k \leq n$ components has at least $n - k$ edges.
2. Prove that in a *directed complete* graph $G = \langle V, E \rangle$, with $|V| = n > 0$,

$$\boxed{\sum_{v \in V} \delta^+(v)^2 = \sum_{v \in V} \delta^-(v)^2}.$$

3. A *casual walk* of length $n \geq 0$ on an undirected graph is a sequence σ of the form $v_0 \xrightarrow{e_1} v_1 \xrightarrow{e_2} \cdots \xrightarrow{e_n} v_n$, where not all edges or vertices are distinct. σ is a *closed* casual walk of length n if $v_0 = v_n$.
   (a) Show that in a tree any closed casual walk is of *even* length.
   (b) Prove that any closed casual walk of *odd* length in an undirected graph contains a cycle (in which all edges are distinct).
4. Prove that any connected undirected graph with $2k$ vertices of odd degree may be decomposed into k *edge-disjoint* subgraphs such that each subgraph has an Euler path.
5. In an old mansion which has only a single entrance, there is a ghost in every room which has an even number of doors. The ghosts are harmless but still scary. Prove that any visitor to the mansion can eventually find a room in which there are no ghosts.
6. Consider Euler graphs which have Hamiltonian circuits. An example of an Euler graph of $n \geq 3$ nodes which also has a Hamiltonian circuit is the graph which looks like an n-sided polygon (may also contain "diagonals" so as to give every vertex an even degree). This is the trivial case.
   (a) Construct a non-trivial undirected Euler graph of at least 5 vertices and at least 7 edges such that the graph has a Hamiltonian circuit. If it is impossible prove that such a graph cannot exist. Otherwise name all the vertices and edges and specify the Eulerian circuit and the Hamiltonian circuit.
   (b) Is it possible to construct non-trivial Euler graphs of n vertices and n edges whose Eulerian circuit is also a Hamiltonian circuit? Prove your case.
7. Let $d(u, v)$ denote the distance of node v from node u in a tree $T = (V, E)$. For any node $u \in V$, $M(u) = max\{d(u, v) \mid v \in V\}$ is the distance of the node that is farthest from u. u is said to be a *centre* of T if $M(u) = min\{M(v) \mid v \in V\}$.
   (a) Prove that every tree has exactly one or two centres.
   (b) One of your colleagues[1] made the following claims. Prove or disprove each claim.
   Claim 1 In any tree with a single centre, every maximal path passes through the centre.
   Claim 2 In any tree with two centres, there is an edge between the two centres and every maximal path contains the edge between the two centres.

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[1]That guy there! No, no, not that one. The other one sitting beside him. That's the one!