Lecture – 10

Solution of Nonlinear Equations - II

Fixed point Problems

Given a function $\boldsymbol{g}:\Re \to \Re$, a value x such that ***x = g(x)*** is called a fixed point of the function ***g***, since ***x*** is unchanged when ***g*** is applied to it. Whereas with a nonlinear equation ***f(x) = 0*** we seek a point where the curve defined by ***f*** intersects the x-axis (i.e., the line y = 0), with a fixed point problem ***x = g(x)*** we seek a point where the curve defined by g intersects the diagonal line ***y = x***.

Many iterative methods for solving nonlinear equations use iteration scheme of form $\boldsymbol{x}_{\boldsymbol{k}+1} = \boldsymbol{g}(\boldsymbol{x}_{\boldsymbol{k}})$, where g is a function chosen so that its fixed points are solutions for f(x) = 0. Such a scheme is called fixed-point iteration or some times functional iteration, since function g is applied repeatedly to initial starting point $\boldsymbol{x}_{\boldsymbol{k}}$.

For a given equation f(x) = 0 , there may be many equivalent fixed-pint problems x= g(x) with different choices for function g. But not all fixed-point formulations are equally useful in deriving an iteration scheme for solving a given nonlinear equation. The resulting iteration schemes may differ not only in their convergence rates but also in whether they converge at all.

Convergence of Fixed-Point Iteration

The behavior of fixed-point iteration schemes can vary widely, from divergence, to slow convergence, to rapid convergence. What makes the difference?

If x* = g(x*) and $|\boldsymbol{g}'(\boldsymbol{x}^*)| < 1$, then the iterative scheme is locally convergent, i.e. there is an interval containing x* such that fixed-point iteration with g converges if started at a point within that interval. If $|\boldsymbol{g}'(\boldsymbol{x}^*)| > 1$, on the other hand, then fixed-point iteration with g diverges for any starting point other than x*.

An iterative method is said to be or order ‘r’ or has the rate of convergence ‘r’, if ‘r’ is the largest positive real number for which there exits a finite constant $\boldsymbol{C} \neq 0$ such that

$$|\varepsilon_{\boldsymbol{k}+1}| \leq \boldsymbol{C}|\varepsilon_{\boldsymbol{k}}|^{\boldsymbol{r}}$$

where $\varepsilon_{\boldsymbol{k}} = \boldsymbol{x}_{\boldsymbol{k}} - \boldsymbol{x}^*$ is the error in the ***kth*** iteration. C is the asymptotic error constant usually depends on the derivatives of ***f(x)*** at ***x = x****. ***x**** is the true solution.

Iterative methods

Bisection method makes no use of the function value other than their sign, which results in slow but sure convergence. Using the function values by iterative methods can derive more rapidly converging methods.

Iterative methods based on first-degree equation

Let f(x) = 0 is a nonlinear equation, Thus, if we approximate f(x) by a first degree equation in the neighborhood of the root the we may write $f(x) = a_0 x + a_1 = 0$. The solution of this is given by $x_1 = -\frac{a_1}{a_0}$, where $a_0 \neq 0$ and a_1 are parameters to be determined by prescribing two appropriate conditions on ***f(x)*** and/or its derivatives.

Newton-Raphson Method

We determine a_0 and a_1, using the condition

$$f_k = a_0 x_k + a_1 \Rightarrow a_1 = -a_0 x_k + f_k \text{ and } \frac{df_k}{dx_k} = f_k' = a_0$$

Thus, $x = -\frac{a_1}{a_0}$ gives $x_{k+1} = -\frac{-a_0 x_k + f_k}{a_0} = x_k - \frac{f_k}{f_k'}$ i.e. the Newton-Raphson iteration is:

$$x_{k+1} = x_k - \frac{f_k}{f_k'}$$

Geometric representation

Newton's method approximates nonlinear function f near x_k by tangent line at $f(x_k)$.

Or

In the limit when $x_{k+1} \to x_k$, the chord passing through the points (x_k, $f(x_k)$) and (x_{k-1}, $f(x_{k-1})$) becomes the tangent at the point (x_k, $f(x_k)$).

Algorithm

x_0 = Initial guess
for k = 0, 1, 2, 3, ….

$$x_{k+1} = x_k - f(x_k) \big/ f'(x_k)$$

end

Note: It requires two function evaluations f_k and f_k' per iteration.

Example: Use Newton's method to find root of $f(x) = x^2 - 4\sin(x) = 0$

$f'(x) = 2x - 4\cos(x)$, Thus the Newton's iteration:

$$x_{k+1} = x_k - \frac{x_k^2 - 4\sin(x_k)}{2x_k - 4\cos(x_k)}, \text{ Take } x_0 = 3$$

Convergence Analysis

Let x* is the exact solution. ε_k = Error at ***kth*** iteration = $x_k - x^*$. Substitute the values of x_k and x_{k+1} in the Newton's iteration formula:

$$x_{k+1} = x_k - f(x_k)\big/f'(x_k), \text{ we get}$$

$$x^* + \varepsilon_{k+1} = x^* + \varepsilon_k - \frac{f(x^* + \varepsilon_k)}{f'(x^* + \varepsilon_k)}$$

Expand by Taylor series about the point x*, we get

$$\varepsilon_{k+1} = \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}\varepsilon_k^2 + O(\varepsilon_k^3)$$

on neglecting ε_k^3 and higher powers of ε_k, we get

$$\varepsilon_{k+1} = C\varepsilon_k^2, \text{ where } C = \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}$$

Here, the rate of convergence r = 2. Hence Newton's method has second order convergence. Another way of stating this is that the number of correct digits in approximate solution is doubled at each iteration of Newton's method.

Remarks: 1. For multiple root, Newton's method is linearly convergent, with asymptotic constant C = (1-1/m), where m is the multiplicity of the root. For example:

k	$f(x) = x^2 - 1$	$f(x) = x^2 - 2x + 1$
0	2.0	2.0
1	1.25	1.5
2	1.025	1.25
3	1.0003	1.125
4	1.00000005	1.0625
5	1.0	1.03125

2. Caution: these convergences are local and hence if starting point is far from solution, method may not converge. e.g. A relatively small value for $f'(x_k)$ (i.e. a nearly horizontal tangent) tends to cause the next iterate to lie far away from the correct approximation.

3. One drawback of Newton's method is, it requires evaluation of both function and its derivative at each iteration.

Secant Method

The derivative may be inconvenient or expansive to evaluate, so we might consider replacing it by a finite difference approximation using some small step size 'h':

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

Thus, the Secant method is: $x_{k+1} = x_k - f(x_k)\dfrac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$

Algorithm

x_0, x_1 = Initial guesses

for k = 0, 1, 2, 3, ….

$$x_{k+1} = x_k - f(x_k)\frac{x_k - x_{k-1}}{f(x_k) - f(x_{k-1})}$$

end

Geometric representation

Approximating the function by the secant line through the previous two iterates, and taking the zero of the resulting linear function to be the next approximate solution.

Regula-Falsi Method

This uses the same iterative formula as Secant method and if the approximations are such that $f_k \cdot f_{k-1} < 0$, it's called Regula-Falsi method.

Remark: 1. Since $(x_k, f(x_k))$ and $(x_{k-1}, f(x_{k-1}))$ are known before the start of the iteration, the Secant method requires one function evaluation per step.

Convergence Analysis

Let x* be a simple root of f(x) = 0 and substitute $x_k = x^* + \varepsilon_k$ in iterative formula, we get

$$\varepsilon_{k+1} = \varepsilon_k - \frac{(\varepsilon_k - \varepsilon_{k-1})\, f(x^* + \varepsilon_k)}{f(x^* + \varepsilon_k) - f(x^* \varepsilon_{k-1})}$$

Using Taylor series expansion about x* and noting that f(x*) = 0, we obtain,

$$\varepsilon_{k+1} = \varepsilon_k - \left[\varepsilon_k + \frac{1}{2}\varepsilon_k^2 \frac{f''(x^*)}{f'(x^*)} + \cdots\right]\left[1 + \frac{1}{2}(\varepsilon_{k-1} + \varepsilon_k)\frac{f''(x^*)}{f'(x^*)}\right]^{-1}$$

or, $\varepsilon_{k+1} = \frac{1}{2}\frac{f''(x^*)}{f'(x^*)}\varepsilon_k\varepsilon_{k-1} + O(\varepsilon_k^2\varepsilon_{k-1} + \varepsilon_k\varepsilon_{k-1}^2)$

or, $\varepsilon_{k+1} = C\varepsilon_k\varepsilon_{k-1}$: Error Equation, where C = $\frac{1}{2}\frac{f''(x^*)}{f'(x^*)}$

By the definition of the convergence $\varepsilon_{k+1} = A\varepsilon_k^r$, where A and r to be determined.

This also gives $\varepsilon_k = A\varepsilon_{k-1}^r$ and $\varepsilon_{k-1} = A^{-1/r}\varepsilon_k^{1/r}$. Substitution of these values, we get $\varepsilon_k^r = CA^{-(1+1/r)}\varepsilon_k^{1+1/r}$, Comparing the power of ε_k on both sides, we get

r = 1 + 1/r, which implies $r = \frac{1}{2}\left(1 \pm \sqrt{5}\right)$, neglecting minus sign gives **r = 1.618 .** Thus Secant method has superlinear rate of convergence.