Short Length Menger's Theorem and Reliable Optical Routing

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ABSTRACT

We deal with a generalization of the Minimum path colouring problem, *k-Edge disjoint path systems colouring*: given a graph G and a set of pairs of vertices of G, the task is to connect each pair by a system of k-edge disjoint paths (a *k-system*) and to colour the k-systems by minimal number of colours in such way that any two edge-intersecting k-systems have different colours. Multiple connecting paths between the same pair of vertices are motivated by a need for fault tolerant connections. We propose an $O(k^2F)$ approximation algorithm for this problem where F is the flow number of the graph. As a byproduct of our analysis we also show that any two k-connected vertices in G are connected by k edge disjoint paths of average length $O(kF)$ which improves previously known bounds for many classes of graphs.

Categories and Subject Descriptors

G.2.2 [**Graph Theory**]: Path and circuit problems, Network problems; F.2.2 [**Nonnumerical Algorithms and Problems**]: Routing and Layout

General Terms

Algorithms

Keywords

Menger's theorem, fault-tolerant routing, path coloring, Flow Number

1. INTRODUCTION

In this paper we describe both efficient and reliable (fault-tolerant) algorithms for network communication problems. We consider optical networks in which faults may appear on one or many links (edges). We guarantee fault tolerance by reserving several independent communication channels (i.e. disjoint paths) for a single connection.

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Formally, we consider a generalization of the well known Minimum path colouring problem (MPCP) [4], namely the **k-Edge disjoint path systems colouring** problem (k-EDPCOL): given a graph $G = (V, E)$ and a (multi)set of pairs of vertices $\{(s_i, t_i)|s_i, t_i \in V\}$, connect each pair by k-edge disjoint paths (called a *k-system*) and colour the k-systems in such a way that any two k-systems using the same edge have different colours. The objective is to minimize the number of colours used.

Since MPCP, which corresponds to 1-EDPCOL, is an NP-hard problem, we concentrate on approximation algorithms for k-EDPCOL. We parametrize the performance of our algorithms by the flow number F of the graph, defined below. This makes it possible to do better than the $\Omega(m^{1/2-\epsilon})$ lower bound [2] for many classes of graphs (we recall that the flow number of, e.g., constant degree expanders, hypercubes, butterflies is $O(\log n)$, $F(d\text{-dim-mesh}) = O(\sqrt[d]{n})$).

The main algorithmic result in this paper is an $O(k^2F)$ approximation algorithm for k-EDPCOL. This improves the best known result for MPCP by a $\log n$ factor. Our algorithm takes an instance of k-EDPCOL, solves it fractionally as a multicommodity flow problem, rounds the fractional flows into integral k-systems and colours the k-systems greedily. Keeping the k-systems small is the crux of our technique (the *size* of a k-system is the number of edges in it). A corollary of this effort is the short length Menger's theorem, namely, an $O(kF)$ bound on the average path length of k edge disjoint paths between any two k-connected vertices.

We conclude this introduction with a formal definition of the flow number. Let I_0 denote an instance of the concurrent multicommodity flow problem with a commodity of demand $\deg(u) \cdot \deg(v)/2|E(G)|$ between each pair of nodes (u, v). For a feasible solution $\mathcal{S}$, let $D(\mathcal{S})$ be the length of the longest flow path in $\mathcal{S}$ and let $C(\mathcal{S})$ be the inverse of the flow. The *flow number* $F = F(G)$ of G is the minimum of $\max\{D(\mathcal{S}), C(\mathcal{S})\}$ over all feasible solutions $\mathcal{S}$ of I_0 [3]. F is known to be $O(\Delta\alpha^{-1} \log n)$ [3], where Δ is the maximum degree of G and α is the edge expansion.

2. A SHORT LENGTH VERSION OF MENGER'S THEOREM

The key problem we address in this paper is the existence of small k-systems, and our ability to find them. For a single pair, this can be easily done by solving a minimum cost flow problem. For multiple pairs, however, we need the following general lemma:

LEMMA 2.1. *Given a graph with flow number F, a set T of pairs of vertices and a flow system $\mathcal{F}$ such that there are k units of flow between each pair from T and the maximal flow through an edge is one, then there exists a flow system $\mathcal{F}'$ such that*

- *there are still k units of flow for each pair from T,*
- *the maximal flow through any edge is 2,*
- *for each pair from T, the corresponding flow can be decomposed into flow along several k-systems, each of size $O(k^2F)$.*

An important tool for proving the lemma is the Shortening lemma [3]. Applying Lemma 2.1 to a single pair of k-connected vertices yields the following bounded length version of Menger's theorem. Since graphs with diameter $\Omega(F)$ are known [3], the result is, up to a factor k, the best possible for general graphs:

COROLLARY 2.2. *Given a graph G with flow number F and a pair of k-connected vertices (u, v), there are k-edge disjoint paths between u and v of average length $16k \cdot F$.*

Lemma 2.1 also implies an improvement of the approximation ratio for the k-edge disjoint paths problem, cf. [1].

3. AN ALGORITHM FOR K-EDPCOL

Let $T \subseteq V \times V$ be an instance of k-EDPCOL on a graph $G(V, E)$. For each $(a_j, b_j) \in T$ let $\mathcal{P}_j$ denote the set of all k-systems of size at most $16k^2F$ between a_j and b_j and let $\mathcal{P} = \bigcup \mathcal{P}_j$. In the following linear program we have a variable $x(s)$ for each k-system s from $\mathcal{P}$ denoting $1/k$th of the total flow sent along the k-system (that is, there is a flow $x(s)$ along each of the k paths of s) and a variable c denoting the maximal edge congestion in the network:

$$\min c \tag{1}$$

$$\sum_{s \in \mathcal{P}: e \in s} x(s) \leq c \quad \forall e \in E \tag{2}$$

$$\sum_{s \in \mathcal{P}_j} x(s) \geq 1 \quad \forall j \tag{3}$$

$$x(s) \geq 0 \quad \forall s \in \mathcal{P} \tag{4}$$

LEMMA 3.1. *The optimal (fractional) solution to the linear program is (up to a constant multiple) a lower bound for the optimal (integral) solution to k-EDPCOL.*

PROOF. Consider the optimal solution to the k-EDPCOL. This is an integral solution to the above LP. We shorten the k-systems for each colour separately. Since the k-systems of the same colour are disjoint, it is possible to apply Lemma 2.1 to shorten them. Taking the modified flows for all colours together we get a fractional solution consisting of only small k-systems and congestion that is guaranteed to be at most $O(1)$ times worse than the original optimal integral. □

THEOREM 3.2. *For any graph G there is an algorithm for k-EDPCOL with approximation ratio $O(k^2F)$, where F is the flow number of G.*

The structure of the algorithm is as follows:

1. (Approximately) solve the linear program (1)-(4). By Lemma 3.1, the congestion C in the approximate solution is (roughly) a lower bound on the number of colours needed in the optimal integral solution.
2. Round the fractional solution to an integral one using an iterative rounding method. The aim of the process will be to guarantee that every k-system intersects with at most $O(Ck^2F)$ other k-systems at the end.
3. The resulting k-systems can be coloured with $O(Ck^2F)$ colours which is within an $O(k^2F)$ factor of the optimum.

Step 1 can be solved either by LP methods or by more efficient approximation algorithms for multicommodity flow problems (constant approximations are sufficient for us). The rounding of Step 2 can be done in a sequence of rounds and requires some effort, cf. [5] (for more than $m \log m$ requests, a standard randomized rounding can be applied). The last step can be done using greedy colouring, it follows easily from Vizing's theorem.

4. CONCLUSION AND OPEN PROBLEMS

We gave an $O(k \cdot F)$ upper bound on the average path length between two k-connected vertices in a graph with flow number F. This compares well with the trivial lower bound $\Omega(F)$. We feel that the gap is an artifact of the analysis and that it can be removed. This would lead to an $O(kF)$ approximation ratio for the k-EDPCOL. A challenging problem left open is that of finding k-systems and assigning colours to them online: Is there an online algorithm with $O(k^2F)$, or even $O(kF)$, competitive ratio?

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