CSL866: Percolation Theory and Random Graphs
I semester, 2007-08

Minor I

Due: In class on 6th September 2007

1. The FKG inequality is somewhat more general than stated in class. Let us try and understand it in generality and apply it to a setting different from percolation. We start by introducing some notation.

A *lattice* is a partially ordered set in which each pair of elements, x and y, has a unique minimal upper bound, called the *join* of x and y, denoted $x \vee y$, and a unique maximal lower bound, called the *meet* of x and y and denoted $x \wedge y$. A lattice L is said to be *distributive* if, for all $x, y, z \in L$

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

Given a distributive lattice L, a function $\mu : L \to \mathbb{R}_+$ is called *log-supermodular* if

$$\mu(x) \cdot \mu(y) \leq \mu(x \vee y) \cdot \mu(x \wedge y)$$

for all $x, y \in L$.

A function $f : L \to \mathbb{R}$ is *non-decreasing* if $f(x) \leq f(y)$ whenever $x \leq y$ and *non-increasing* if $f(x) \geq f(y)$ whenever $x \leq y$.

In this setting, we have the following statement of the FKG inequality:

Theorem 1.1 (FKG Inequality) *Let L be a finite distributive lattice and let $\mu : L \to \mathbb{R}_+$ be a log-supermodular funtion. Then if $f, g : L \to \mathbb{R}_+$ are both non-decreasing or non-increasing, we have*

$$\left(\sum_{x\in L} \mu(x) f(x)\right) \cdot \left(\sum_{x\in L} \mu(x) g(x)\right) \leq \left(\sum_{x\in L} \mu(x) f(x) g(x)\right) \cdot \left(\sum_{x\in L} \mu(x)\right)$$

If we take μ to be a measure on L. Assuming μ to not be identically 0, we can define the expectation w.r.t. μ of a function f as

$$\mathrm{E}_\mu[f] = \frac{\sum_{x\in L} \mu(x) f(x)}{\sum_{x\in L} \mu(x)}.$$

With this notation, we can restate the FKG inequality as follows: For any log-supermodular funtion μ defined on a distributive lattice and any two functions $f : L \to \mathbb{R}$ and $g : L \to \mathbb{R}$ which are both non-increasing or non-decreasing,

$$\mathrm{E}_\mu[f \cdot g] \geq \mathrm{E}_\mu[f] \cdot \mathrm{E}_\mu[g].$$

And, if one of them is non-increasing and the other non-decreasing,

$$\mathrm{E}_\mu[f \cdot g] \leq \mathrm{E}_\mu[f] \cdot \mathrm{E}_\mu[g].$$

Let us now try and apply this to a simple example.

Suppose we throw m balls independently into n bins uniformly at random, for positive integers m and n. Let X_i be the random variable denoting the number of balls in the ith bin.

We use an m-dimensional vector $\hat{a} = (a_1, a_2, \ldots, a_m)$ to denote each outcome of the experiment, where a_i is the bin number (between 1 and n) of the bin into which the ith ball landed. Let L be the set of all such outcomes.

Define a partial order: $\hat{a} \leq_L \hat{b}$ if $a_i \leq b_i$ for all $i \in [m]$.

Problem 1 *Argue that L under the partial order $\leq_L$ forms a distributive lattice.*

Now, define the measure $\mu : L \to \mathbb{R}$ by $\mu(\hat{a}) = 1/n^m$ for every $\hat{a} \in L$.

Problem 2 *Argue that μ is log-supermodular.*

Now, we are in a position to use the FKG inequality to show the following result:

Problem 3 *Now, given two bins, i and j, prove that for any two positive integers $t_i, t_j \leq m$,*

$$Pr[(X_i \geq t_i) \wedge (X_j \geq t_j)] \leq Pr[X_i \geq t_i] \cdot Pr[X_j \geq t_j]$$

Use the result of Problem 3 and show the following:

Problem 4 *X_i and X_j are negatively correlated.*

2. Consider the network G depicted in Figure 1. There are n nodes from s to t. Each node is connected to its two neighbours by $\log n$ paralell edges.

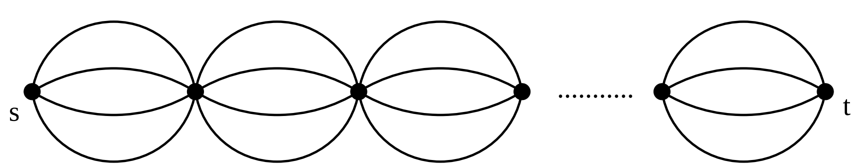

Figure 1: A multi-edged network

Now suppose each edge is removed with probability 1/2, to give a network G'.

Problem 5 *Prove that the min cut between s and t in G' is at most $\log n/2$ with probability $1 - \theta(\frac{1}{n^\epsilon})$ for some non-negative ϵ. What is the value of ϵ?*

A brute force approach to this problem will probably be very hard. The best way to approach it would be to use the tools developed in the lecture. An additional hint is that looking up Menger's theorem might help you.

3. A cow is standing in front of an infinite fence (see Figure 2). On the other side is grass. The cow wants to get to this grass. Somewhere along this fence is a hole through which the cow can get to the other side. The distance d from the cow to the hole has a probability distribution $f(d)$ associated with it i.e. the probability that the hole is k steps away from the cow is given by $f(k)$. Note that we think of all distances as discrete i.e. they are always measured in terms of steps taken by the cow. The cow can take negative integer steps as well as positive integer steps, i.e. k steps to the left and steps to the right respectively. Also, we know that

$$\sum_{k=-\infty}^{\infty} |k| \cdot f(k) < \infty.$$

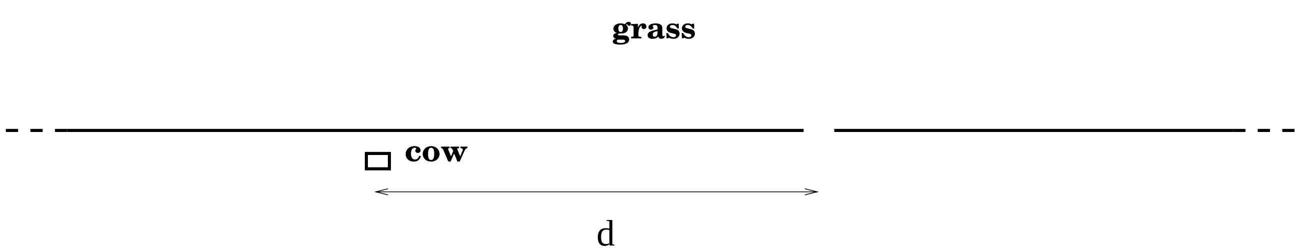

Figure 2: A cow at the fence

We want to help the cow by describing an algorithm that can find the hole with probability 1.

Problem 6 *What is a sufficient condition for an algorithm to be able to find the hole with probability 1?*

Problem 7 *Describe such an algorithm.*