CSL866: Percolation Theory and Random Graphs
I semester, 2007-08

Major exam

Due: In my mailbox by 11:59 AM on Wednesday, 28th November 2007

Examination notes: Please do not discuss the exam with anyone or look up any book or the internet. The only materials you are allowed to consult are the class notes and the paper by Alon, Benjamini and Stacey.

1. For bond percolation in 2-dimensions, if p is such that $p_c < p < 1$, show that there exists an $\eta(p) > 0$ such that the probability of the cluster containing the origin has the following property
$$\mathrm{P}_p(|C| = n) \leq \exp(\eta(p)\sqrt{n}).$$
Hint. Use duality, and the fact that any open cluster of size exactly n must have a circuit of closed edges of the dual of size at least $\lambda\sqrt{n}$ around it for some constant $\lambda > 0$.

2. In the supercritical phase define $\tau_p^f(x, y)$ to be the probability that vertices x and y are part of the same *finite* open cluster. Also, the vertex $e_n = (n, 0, \ldots)$ is the vertex which lies at distance n on the x_1 axis. Show that there is a $\sigma(p) > 0$ and $A(p, d)$ as given in Theorem 10.1 (Lecture 10) such that
$$\lim_{n\to\infty} \left\{ -\frac{1}{n} \log \tau_p^f(0, e_n) \right\} = \sigma(p)$$
and, for all n,
$$\tau_p^f(0, e_n) \leq A(p, d) n^d e^{-n\sigma(p)}.$$
Hint. For a lower bound on $\tau_p^f(0, e_n)$, consider $n = 2m + 2$ and come up with a way of putting together a cluster which contains both 0 and e_{2m+2} from two clusters of diameter m.