Lecture 8: The subcritical phase: Asymptotic tail behaviour of the radius of an open cluster

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In this lecture, we go back to the problem of looking at the probability of the origin being connected to the surface of the box $B(n)$ i.e. $\{0 \leftrightarrow \partial B(n)\}$. Earlier, in Lecture 4, we gave an upper bound for this probability. Now we characterize it more fully, giving both lower and uppper bounds.

In order to do this, we will introduce another tool from analysis first.

8.1 Preliminaries: The Subadditive Limit Theorem

A sequence $(a_i : i \geq 1)$ is called *subadditive* if $a_{m+n} \leq a_m + a_n$ for all $m, n \geq 1$. A sequence $(a_i : i \geq 1)$ is called *superadditive* if $(-a_i : i \geq 1)$ is subadditive.

Some simple sequences that are subadditive are $a_i = \sqrt{i}$ and $b_i = i$, the last one being both subadditive and superadditive.

Theorem 8.1 *If a sequence $(a_i : i \geq 1)$ is subadditive, then the limit*

$$\lambda = \lim_{r \to \infty} \frac{a_r}{r}$$

exists and $-\infty \leq \lambda < \infty$. Additionally,

$$\lambda = \inf \left\{ \frac{a_m}{m} : m \geq 1 \right\}.$$

The fact that the limit of the sequence λ is the $\inf\{\frac{a_m}{m} : m \geq 1\}$ is important as it allows us to lower bound each element of the sequence: ,

$$\forall m \geq 1 : a_m \geq m\lambda.$$

8.2 Asymptotic tail behaviour of radius of an open cluster

We will try to show that as n increases the probability of the event $\{0 \leftrightarrow \partial B(n)\}$ is not just upper bounded by an exponential decay but is also lower bounded similarly. In this lecture we will prove the following theorem:

Theorem 8.2 *For any p such that $0 < p < 1$, there exist strictly positive constants ρ and σ, independent of p, and a function $\varphi(p)$, such that*

$$\rho n^{1-d} e^{-n\varphi(p)} \leq P_p(0 \leftrightarrow \partial B(n) \leq \sigma n^{d-1} e^{-n\varphi(p)}.$$

Proof. Let us call $\beta(n) = \mathrm{P}_p(0 \leftrightarrow \partial B(n))$. Our aim will be to show that the sequence $\log \beta(n)$ is both subadditive and superadditive up to small additive factors. Once we have done this, the subadditive limit theorem will give us the proof.

Let us begin by proving subadditivity.

Lemma 8.3 *For $m, n \geq 1$,*

$$\beta(m+n) \leq |\partial B(m)| \cdot \beta(m) \cdot \beta(n).$$

Proof for Lemma 8.3. The core of the proof of this lemma has been discussed in Lecture 4 as part of the proof of Theorem 4.3. We repeat it here for completeness and to demonstrate the relationship to the proof of the next lemma.

Consider $\beta(m+n)$, the probabilty of the existence of an open path from origin to $\partial B(m+n)$. If such a path exists, there must be an $x \in \partial B(m)$ such that the origin is connected to x through an open path and further there is a path disjoint from the path used to connect 0 to x which connects x to $\partial B(n, x)$ on its way to $\partial B(m+n)$. Hence, using the BK inequality

$$\beta(m+n) \leq \sum_{x \in \partial B(m)} \tau_p(0, x) \cdot (\mathrm{P}_p(x \leftrightarrow \partial B(n, x)))$$

Because of translational invariance the events $\{x \leftrightarrow \partial B(n, x)\}$ and $\{0 \leftrightarrow \partial B(n)\}$ are equivalent. So

$$\beta(m+n) \leq \sum_{x \in \partial B(m)} \tau_p(0, x) \cdot \beta(n) \tag{1}$$

Also, because $x \in \partial B(m)$, so the event $\{0 \leftrightarrow x\}$ implies the event $\{0 \leftrightarrow \partial B(m)\}$. And so we have

$$\forall x \in \partial B(m) : \tau_p(0, x) \leq \beta(m).$$

Substituting this in (1), we get

$$\beta(m+n) \leq \sum_{x \in \partial B(m)} \beta(m) \cdot \beta(n) \leq |\partial B(m)| \cdot \beta(m) \cdot \beta(n)$$

■

Corollary 8.4 *For* $m, n \geq 1$*,* $\log \beta(m+n) \leq \log \beta(m) + \log \beta(n) + g(m)$*, where* $g(r) = \log(d^2 3^{d+1}) + (d-1) \cdot \log(r)$

Proof for Corollary 8.4. Consider a face of $\partial B(m)$, say the one for which $x_1 = m$. Then the number of lattice points in it are

$$|\{x \in \mathbb{Z} : x_1 = m, |x_i| \leq m, 2 \leq i \leq d\}| = (2m+1)^{d-1} \tag{2}$$

since there are $(d-1)$ other coordinates, where each coordinate can take any of $2m+1$ different values. As all the 2d faces of $\partial B(m)$ are identical, we get

$$|\partial B(m)| \leq 2d \cdot (2m+1)^{d-1} \leq d \cdot 3^d \cdot m^{d-1} \leq d^2 \cdot 3^{d+1} \cdot m^{d-1} \tag{3}$$

by using equation (2). The above equation is an inequality because the corner points are being counted more than once, as they lie on more than one face of $\partial B(m)$ (d faces to be precise). Taking logs on both sides of the inequality in Lemma 8.3 and applying (3), we get the result. ■

With this corollary we are within sight of subadditivity, although not quite there because of the presence of $g(m)$ on the right hand side. But before we apply subadditivity, let us try to show another result which will help us prove superadditivity,

Lemma 8.5 *For* $m, n \geq 1$

$$\beta(m+n) \geq \frac{\beta(m) \cdot \beta(n)}{2d|\partial\beta(m)|}.$$

Proof for Lemma 8.5. Consider a particular face of $\partial B(m)$, say the face f with $x_1 = m$. Let $\gamma(m) = \mathrm{P}_p(0 \leftrightarrow x : x \in f)$. Since for the origin to be connected to $\partial B(m)$, it must be connected to some face of $\partial B(m)$, and since the probability of the origin being connected to each face is exactly the same as the probability of its being connected to f, we have

$$\beta(m) \leq \sum_{\text{all faces}} \gamma(m) = 2d \cdot \gamma(m). \tag{4}$$

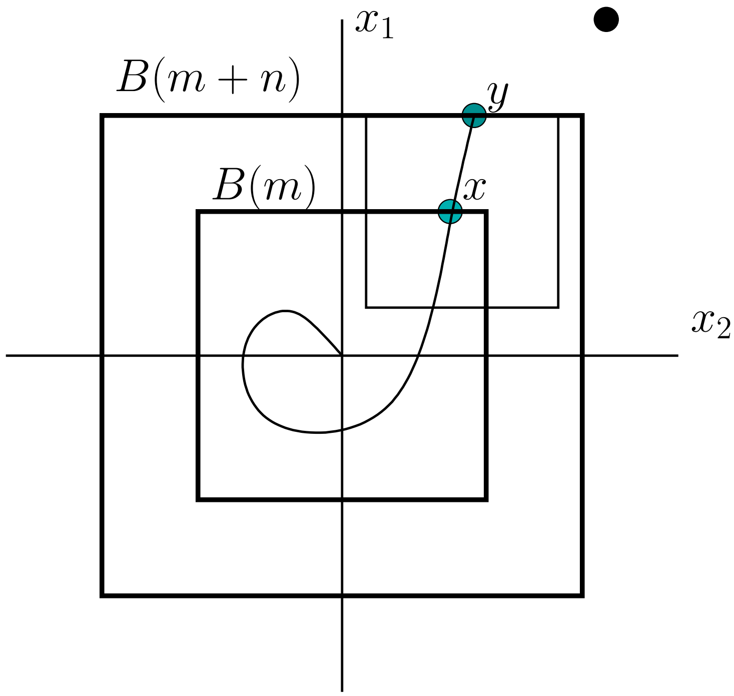

Figure 1: Illustration for Lemma 8.5

Now, consider some $x \in \partial B(m)$. There must be a k such that $x_k = \pm m$ (depending on the face of $\partial B(m)$ x lies on). Let us assume that $x_k = +m$, the case where it is $-m$ will follow similarly.

Denote by U_x the event $\{0 \leftrightarrow x\}$ and by V_x the event that there is an open path in $B(n, x)$ joining x to some vertex $y \in \partial B(n, x)$ for which $y_k = m + n$ i.e. x and y lie on the parallel faces of $\partial B(m)$ and $\partial B(m + n)$.

From Figure 1 it is clear that $U_x \cap V_x \Rightarrow \{0 \leftrightarrow \partial B(m+n)\}$. Hence

$$\beta(m+n) \geq \mathrm{P}_p(U_x \cap V_x).$$

Since U_x and V_x are increasing events, we can use the FKG inequality:

$$\beta(m+n) \geq \mathrm{P}_p(U_x) \cdot \mathrm{P}_p(V_x). \tag{5}$$

Let us consider the two terms on the right hand side of this inequality one at a time. Firstly, since the event V_x is defined in terms of a particular face of $\partial B(n,x)$, by translational invariance we get that,

$$\mathrm{P}_p(V(x)) = \gamma(n).$$

For the second term, we see that by the definition of $\beta(m)$, we get that

$$\beta(m) = \mathrm{P}_p\left(\bigcup_{x \in \partial B(m)} U(x)\right) \leq \sum_{x \in \partial B(m)} \mathrm{P}_p(U(x)).$$

By the Pigeon-hole Principle, we can say that there is an x^* such that

$$\mathrm{P}_p(U(x^*)) \geq \frac{1}{|\partial B(m)|}\beta(m).$$

Since (5) holds for this choice of x^* as well, we can now say that

$$\begin{aligned}\beta(m+n) &\geq \mathrm{P}_p(U(x^*)) \cdot \mathrm{P}_p(V(x^*)) \\ &\geq \frac{1}{|\partial B(m)|}\beta(m)\gamma(n)\end{aligned}$$

Using equation (4) to replace $\gamma(n)$ we finally get

$$\beta(m+n) \geq \frac{\beta(m)\beta(n)}{2d|\partial B(m)|}$$

■

Corollary 8.6 *For $m, n \geq 1$,* $\log \beta(m+n) \geq \log \beta(m) + \log \beta(n) - g(m)$, *where* $g(r) = \log(d^2 3^{d+1}) + (d-1) \cdot \log(r)$

Proof for Corollary 8.6. Taking log on both sides of in the statement of Lemma 8.5 and using equation (3), we get the result. ∎

Corollary 8.4 and Corollary 8.6 show us that $\log \beta(n)$ is close to being subadditive and superadditive but is not quite there. We need to massage the inequalities we have got to get the required property.

Let us begin with Corollary 8.4. Adding $g(n)$ to both sides we get

$$\log \beta(m+n) + g(n) \leq \log \beta(m) + g(m) + \log \beta(n) + g(n) \tag{6}$$

Also, if $m \leq n$,

$$g(m+n) - g(n) = (d-1)\log(1 + \frac{m}{n}) \leq (d-1)\log 2 \tag{7}$$

Adding equations (6) and (7) and adding $(d-1)\log 2$ on both sides, we get

$$\begin{aligned}\log \beta(m+n) + g(m+n) + (d-1)\log 2 \quad \leq \quad & (\log \beta(m) + g(m) + (d-1)\log 2) \\ & + (\log \beta(n) + g(n) + (d-1)\log 2.\end{aligned}$$

Thus, even though the sequence $\beta(n)$ is not subadditive, the sequence $a_k = \log \beta(k) + g(k) + (d-1)\log 2$ is. And so by the Subadditive Limit Theorem we get that the limit

$$\lambda = \lim_{k\to\infty} \left(\frac{\log \beta(k)}{k} + \frac{g(k)}{k} + \frac{(d-1)\log 2}{k} \right) \tag{8}$$

exists. We can see that both $\frac{g(k)}{k}$ and $\frac{(d-1)\log 2}{k}$ tend to 0 as $k \to \infty$. Therefore, it's safe to say that $\varphi(p) = -\lim_{k\to\infty} \frac{\log \beta(k)}{k} = -\lambda$ exists. From the Subadditive Limit Theorem, we also have that

$$\log \beta(n) + g(n) + (d-1)\log 2 \geq n\lambda = -n \cdot \varphi(p)$$

Hence

$$\beta(n) \geq e^{-n\phi(p)} \frac{1}{d^2 3^{d+1} n^{d-1}} \cdot \frac{1}{2^{d-1}}$$

Setting $\rho = \frac{1}{d^2 3^{d+1}} \cdot \frac{1}{2^{d-1}}$, we get

$$\beta(n) \geq \rho \cdot n^{1-d} \cdot e^{-n\phi(p)} \tag{9}$$

We proceed in a similar fashion for the proof of the upper bound, using the result from Corollary (8.4) instead of Corollary (8.6) and subtracting $g(n)$ from both sides instead of adding it. As a result, equation (6) becomes

$$\log \beta(m+n) - g(n) \geq \log \beta(m) - g(m) + \log \beta(n) - g(n) \tag{10}$$

and equation (7), multiplied by -1, becomes

$$g(n) - g(m+n) = -(d-1)\log(1+\frac{m}{n}) \geq -(d-1)\log 2 \tag{11}$$

if $m \leq n$. Adding equations (10) and (11) and subtracting $(d-1)\log 2$ from both sides, we get

$$\begin{aligned} \log \beta(m+n) - g(m+n) - (d-1)\log 2 \quad &\geq \quad (\log \beta(m) - g(m) - (d-1)\log 2) \\ &\quad + (\log \beta(n) - g(n) - (d-1)\log 2). \end{aligned}$$

This time around, we can say that the sequence $-b_k = -\log \beta(k) + g(k) + (d-1)\log 2$ is subadditive. Furthermore, from Theorem (8.1), we again get that the limit

$$\lambda = \lim_{k\to\infty} \left(\frac{-\log \beta(k)}{k} + \frac{g(k)}{k} + \frac{(d-1)\log 2}{k} \right) \tag{12}$$

exists, where both $\frac{g(k)}{k}$ and $\frac{(d-1)\log 2}{k} \to 0$ as $k \to \infty$.As a result, $\varphi(p) = -\lim_{k\to\infty} \frac{\log \beta(k)}{k}$ exists in this case as well.

And so, as before, we get

$$-\log \beta(n) + g(n) + (d-1)\log 2 \geq n \cdot \phi(p)$$

Which gives us

$$\beta(n) \leq e^{-n\phi(p)}(d^2 3^{d+1} n^{d-1})(2^{d-1})$$

Setting $\sigma = (d^2 3^{d+1}) \cdot (2^{d-1})$, we get

$$\beta(n) \leq \sigma \cdot n^{d-1} \cdot e^{-n\phi(p)}$$

$\blacksquare$