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Lecture 7: Exponential decay of the cluster size distribution

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7.1 Introduction

We have seen how the radius of an open cluster decreases exponentially in the subcritical phase ($p < p_c$) as n increases. Apart from implying that the mean cluster size is finite in the subcritical phase i.e. $\chi(p) < \infty$), this also implies an exponential decay in the distribution of the cluster size. Since a radius of n implies a cluster size of $\theta(n^d)$ in $\mathbb{Z}^d$ we can say that the exponential decay of the radius of the open cluster implies a result of the form

$$\mathrm{P}_p(|C| > n^d) \leq e^{-f(p)n}.$$

for some function f with $f(p) > 0$ for all $0 < p < p_c$.

However, we can improve this result significantly to the result stated in Theorem 7.1. The proof of this will be the subject of this lecture.

7.2 Exponential decay of cluster size distribution

Theorem 7.1 *If $0 < p < p_c$, there exists $\lambda(p) > 0$ such that*

$$P_p(|C| \geq n) \leq e^{-n\lambda(p)}, \forall n \geq 1.$$

Proof. We will consider a modified version of the moment generating function of $|C|$, $E_p(|C| \exp(t|C|))$ for some $t > 0$.

First, we introduce some notation.

- $I_{\{0 \leftrightarrow x\}}$: Indicator variable for the event $\{0 \leftrightarrow x\}$.
- $\tau_p(x_1, \ldots, x_n) = P_p\{x_1, \ldots, x_n$ are in the same open cluster$\}$.

Now, we can use the indicator variables $I_{\{0\leftrightarrow x\}}$ to say that

$$|C| = \sum_{x\in\mathbb{Z}^d} I_{\{0\leftrightarrow x\}}$$

Hence we get

$$\begin{aligned}
|C|^n &= \left(\sum_{x\in\mathbb{Z}^d} I_{\{0\leftrightarrow x\}}\right)^n \\
&= \sum_{x_1,\ldots,x_n\in(\mathbb{Z}^d)^n} I_{\{0\leftrightarrow x_1\}}\ldots I_{\{0\leftrightarrow x_n\}} \\
\text{Hence, } E_p(|C|^n) &= \sum_{x_1,\ldots,x_n} \tau_p(0,x_1,\ldots,x_n).
\end{aligned}$$

We will also need the following simple result from graph theory.

Lemma 7.2 *Let $G = (V,E)$ be a connected graph and $\emptyset \neq W \subseteq V$. There exists a vertex $w \in W$ such that, in the graph obtained from G by deleting w and all its incident edges, all vertices in $W - \{w\}$ are in the same connected component.*

Proof of Lemma 7.2. Let T be a spanning tree of G and let u be any vertex of G. We can think of T as a tree rooted at u. Assume $|W| \geq 2$, so that at least one of the branches of T contains a vertex in W. Let $u = x_0, x_1, \ldots, x_m$ be the vertices on such a branch listed in increasing order of distance from u in the tree T. Let $w = \max\{x_j : x_j \in W\}$. Clearly, w has the required property. ■

We know that, $E_p(|C|^0) = 1$ and $E_p(|C|) = \chi(p)$. So, the first non-trivial case is $E_p(|C|^2)$. This is the subject of our next lemma.

Lemma 7.3 *For $0 < p < 1$,*

$$E_p(|C|^2) \leq \chi(p)^3.$$

Proof of Lemma 7.3. Take $G = C(x_0)$ and $W = \{x_0, x_1, x_2\}$. By Lemma 7.2 there is a vertex of these three, say x_2 whose removal leaves x_0 and x_1 in the same open cluster. This is equivalent to saying that there

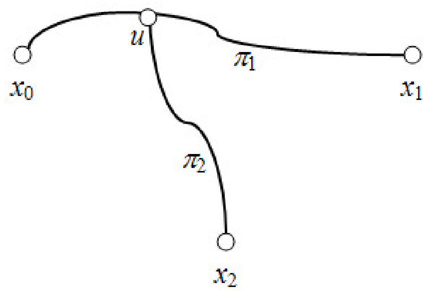

Figure 1: Three edge-disjoint spokes of a wheel.

exists an open path π_1 joining x_0 and x_1 which does not use the third vertex x_2. As $C(x_0)$ is connected, there exists an open path π_2 from x_2 to some vertex u in π_1 using no other vertex in π_1. See Figure 1 for an illustration of this argument. What we have shown is that if three vertices lie in the same open cluster there is a "three spokes in a wheel" structure within that cluster. This will help us use the BK inequality. Note that, though we have assumed that x_0, x_1, x_2 are distinct, this is not required. If $x_0 = x_1 = x_2$, we can take $u = x_0$ and if $x_0 = x_1 \neq x_2$, we can take $u = x_0$.

From the above discussion it follows that,

$$\{x_0 \leftrightarrow x_1 \leftrightarrow x_2\} \subseteq \{\exists u \in \mathbb{Z}^d : \{u \leftrightarrow x_0\} \circ \{u \leftrightarrow x_1\} \circ \{u \leftrightarrow x_2\}\}$$

where $\{x_0 \leftrightarrow x_1 \leftrightarrow x_2\}$ is the event that x_0, x_1 and x_2 are in the same open cluster.

Hence,

$$\begin{aligned}\tau_p(x_0, x_1, x_2) &\leq \sum_{u \in \mathbb{Z}^d} P_p(\{u \leftrightarrow x_0\} \circ \{u \leftrightarrow x_1\} \circ \{u \leftrightarrow x_2\}) \\ &\leq \sum_{u \in \mathbb{Z}^d} \tau_p(u, x_0)\tau_p(u, x_1)\tau_p(u, x_2) \qquad \text{(BK inequality)}\end{aligned}$$

Taking $x_0 = 0$ we get,

$$\begin{aligned} E_p(|C|^2) &\leq \sum_{u,x_1,x_2} \tau_p(u,0)\tau_p(u,x_1)\tau_p(u,x_2) \\ &= \sum_{u\in\mathbb{Z}^d} \tau_p(u,0) \sum_{x_1\in\mathbb{Z}^d} \tau_p(u,x_1) \sum_{x_2\in\mathbb{Z}^d} \tau_p(u,x_2) \\ &= \left(\sum_{x\in\mathbb{Z}^d} \tau_p(x,0)\right)^3 \qquad \text{(Translation invariance)} \\ &= \chi(p)^3. \end{aligned}$$

■

This approach worked well to calculate $|C|^2$. Now let us try and generalize it to calculate $|C|^n$ for $n \geq 3$.

Lemma 7.4 *For $0 < p < 1$,*

$$E_p(|C|^n) \leq \frac{(2n-2)!}{2^{n-1}(n-1)!}\chi(p)^{2n-1}.$$

Proof. To prove this lemma, we have to generalize the idea of "a wheel with three spokes." With this in mind we give the following definitions:

Definition 7.5 *A tree is called a* skeleton *if each of its vertex has degree* 1 *or* 3*. The degree* 1 *vertices are called* exterior *and the degree* 3 *vertices are called* interior.

A labelled skeleton *is one in which each of the k exterior vertices has a unique label between* 0 *to $k-1$.*

Two labelled skeletons S_1 and S_2 are called isomorphic*, if there is a one-one mapping between $V(S_1)$ and $V(S_2)$ under which both the adjacency relation and the labellings of the exterior vertices are preserved.*

Figure 2 shows skeletons with 1, 2 and 3 internal vertices. Note that the smallest possible skeleton (the one with 1 internal vertex) corresponds to a wheel with three spokes and is realized in $\mathbb{Z}^d$ by the structure in Figure 1. Before we show how the argument of the previous lemma can be generalized, let us discuss a simple property of skeletons.

Fact 7.6 *A skeleton with k exterior vertices has $k-2$ interior vertices and $2k-3$ edges.*

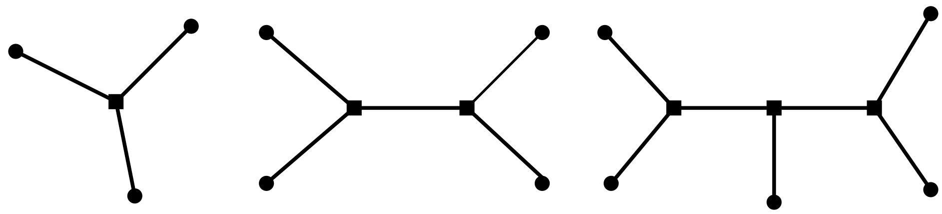

Figure 2: Some simple skeletons. Note that there is only skeleton each with 1, 2 and 3 internal vertices. However, there are more than one skeletons with k internal vertices for $k \geq 4$.

Proof of Fact 7.6. Let x be the number of interior vertices and y be the number of edges. Since, each exterior vertex has degree 1 and each interior vertex has degree 3 and the fact that its a tree, we must have

$$k + 3x = 2(k + x - 1) = 2y$$

Solving this we get, $x = k - 2$ and $y = 2k - 3$. ■

Now, we move on to the general argument. In the previous lemma there was a structure with three disjoint paths realizing the three edges of the skeleton with one internal vertex (the three spokes of the wheel) whenever three vertices were in the same open cluster, we will show that if there are more than three vertices in the same open cluster there will be some larger skeleton whose edges can be realized in the lattice through edge-disjoint paths. We state this argument as a lemma.

Lemma 7.7 *Suppose that $x_0, x_1, \dots, x_k$ are vertices of $\mathbb{L}^d$ belonging to the same open cluster. We claim that there exists a labelled skeleton S with $k+1$ exterior vertices together with a mapping ψ from $V(S)$ to $\mathbb{Z}^d$, such that the following is true:*

1. *The exterior vertex of S with label i is mapped to x_i by ψ, for $0 \leq i \leq k$.*
2. *There exist $2k - 1$ edge-disjoint open paths joining the $2k - 1$ vertex pairs $\{(\psi(u), \psi(v)) : (u, v) \in S\}$.*

Proof of Lemma 7.7. We prove this by induction on k. From the proof of Lemma 7.3, it follows that this is true for $k = 2$. Suppose that it is true for $k = n$ and $x_0, x_1, \dots, x_n$ belong to the same open cluster of $\mathbb{L}^d$. By Lemma

7.2, there exists j such that every vertex in $\{x_i : 0 \leq i \leq n+1\} - \{x_j\}$ belongs to the same connected component of the graph obtained from this cluster by deleting x_j and all its incident edges. We can assume without loss of generality that $j = n+1$. We also assume that all the vertices $x_0, x_1, \ldots, x_n$ are distinct from x_{n+1}. Even without this assumption we can prove this, but we will not go into the details.

By the induction hypothesis, there exists a labelled skeleton S with $n+1$ exterior vertices together with a mapping ψ from $V(S)$ to $\mathbb{Z}^d$ satisfying (1) and (2) with $k = n$, such that x_{n+1} does not lie in any of the edge-disjoint open paths in (2). But, x_{n+1} is in the same open cluster as $x_0, x_1, \ldots, x_n$. Hence, there exists an open path π joining x_{n+1} to some vertex in these $2n-1$ edge-disjoint paths, using exactly one vertex (say z) in these paths. We modify the skeleton S by adding an additional interior vertex v at the appropriate place and by defining $\psi(v) = z$. Therefore, the claim is true for $k = n+1$. ■

It follows that for $k \geq 2$,

$$\tau_p(x_0, \ldots, x_k) \leq \sum_S \sum_\psi P_p(\exists \text{ edge-disjoint paths joining } \psi(u) \text{ to } \psi(v),\ \forall (u,v) \in E(S)).$$

We use the BK inequality to get,

$$\tau_p(x_0, \ldots, x_k) \leq \sum_S \sum_\psi \prod_{(u,v)\in S} \tau_p(\psi(u), \psi(v)).$$

Where the first sum is over all possible labelled skeletons with k external vertices and the second sum is over all possible mappings of a given skeleton.

Hence,

$$\begin{aligned} E_p(|C|^n) &= \sum_{x_1,\ldots,x_n} \tau_p(x_0, \ldots, x_n) \\ &\leq \sum_{x_1,\ldots,x_n} \sum_S \sum_\psi \prod_{(u,v)\in S} \tau_p(\psi(u), \psi(v)) \\ &\leq \left[\sum_x \tau_p(0,x)\right]^n \sum_S \left[\sum_x \tau_p(0,x)\right]^{n-1} \\ &= N_{n+1}\chi(p)^{2n-1}, \end{aligned}$$

where N_{n+1} is the number of labelled skeletons with $n+1$ exterior vertices.

To compute N_{n+1}, we recall that (Fact 7.6) a labelled skeleton with n exterior vertices has $2n-3$ edges. We can attach a new exterior vertex to any of these $2n-3$ edges and add an interior vertex to make it a skeleton on $n+1$ exterior vertices. None of the resulting skeletons are isomorphic to each other. So, we have the following recurrence relation:

$$\begin{aligned} N_{n+1} &= (2n-3)N_n \\ N_3 &= 1 \end{aligned}$$

It follows that,

$$\begin{aligned} N_{n+1} &= (2n-3)(2n-5)\dots 3\cdot 1 \\ &= \frac{(2n-2)!}{2^{n-1}(n-1)!}. \\ E_p(|C|^n) &\leq \frac{(2n-2)!}{2^{n-1}(n-1)!}\chi(p)^{2n-1}. \end{aligned}$$

■

Now, using Lemma 7.4 we have,

$$\begin{aligned} E_p(|C|e^{t|C|}) &= \sum_{n=0}^{\infty} \frac{1}{n!} t^n E_p(|C|^{n+1}) \\ &\leq \chi(p)\left[1+\sum_{n=1}^{\infty}\frac{1}{n!}t^n N_{n+2}\chi(p)^{2n}\right] \\ &= \chi(p)\left(1-2t\chi(p)^2\right)^{-1/2}, \end{aligned}$$

for $0 \leq t < \frac{1}{2\chi(p)^2}$. Using Markov's inequality we get,

$$\begin{aligned} P_p(|C|\geq n) &= P_p(|C|e^{t|C|} \geq ne^{tn}) \\ &\leq \frac{1}{ne^{tn}} E_p(|C|e^{t|C|}) \\ &\leq \frac{\chi(p)}{ne^{tn}}\left(1-2t\chi(p)^2\right)^{-1/2}, \end{aligned}$$

for $0 \leq t < \frac{1}{2\chi(p)^2}$. We choose

$$t = \frac{1}{2\chi(p)^2} - \frac{1}{2n}.$$

Clearly, $t > 0$ if and only if $n > \chi(p)^2$. For this value of t we get,

$$P_p(|C| \geq n) \leq \left(\frac{e}{n}\right)^{1/2} \exp\left(-\frac{n}{2\chi(p)^2}\right).$$

Setting $\lambda(p) = \frac{1}{2\chi(p)^2}$, we arrive at Theorem 7.1. ∎