Lecture 6: The Subcritical phase: Exponential decay of the radius of the open cluster

17th, 24th, 27th September and 1st October 2007

In this lecture we pose the question: when is $\chi(p)$, the expected size of the open cluster containing the origin, finite? Definitely, it is not finite when $p > p_c$. But is it always finite when $p < p_c$? The answer is yes, $\chi(p)$ also undergoes a critical phenomenon at p_c, it is finite below and infinte above. But to give this yes answer requires some work. In this lecture we will prove a stronger result. We will show that the tail probability of the radius of an open cluster decays exponentially. Once we have demonstrated this, the the finiteness of $\chi(p)$ follows.

6.1 Exponential decay of the radius

This section will be devoted to proving the following theorem

Theorem 6.1 *If $p < p_c$, there exists a function $\Psi(p) > 0$, such that*

$$P_p(0 \leftrightarrow \partial S(n)) < e^{-n\Psi(p)}.$$

Proof. Let $g_p(n)$ denote $\mathrm{P}_p(A_n)$. By the Integral form of the Russo's formula(equation(2) of lecture 5), if A_n is an increasing event and $N(A_n)$ denotes the number of pivotal edges for A_n :

$$g_\alpha(n) = g_\beta(n) \exp\left(-\int_\alpha^\beta \frac{E_p(N(A_n)|A_n)}{p} dp\right).$$

Let $0 \leq \alpha < \beta \leq 1$. Since $p < 1$,

$$\frac{E_p(N(A_n)|A_n)}{p} \geq E_p(N(A_n)|A_n).$$

$$g_\beta(n) \exp\left(-\int_\alpha^\beta \frac{E_p(N(A_n)|A_n)}{p} dp\right) \leq g_\beta(n) \exp\left(-\int_\alpha^\beta E_p(N(A_n)|A_n) dp\right).$$

Since $g_\alpha(n) = g_\beta(n) \exp\left(-\int_\alpha^\beta \frac{E_p(N(A_n)|A_n)}{p} dp\right)$, it follows that

$$g_\alpha(n) \leq g_\beta(n) \exp\left(-\int_\alpha^\beta E_p(N(A_n)|A_n) dp\right). \quad (1)$$

To Prove the *Theorem 6.1* , the above inequality will play an important role. For that we need to find $E_p(N(A_n)|A_n)$ so that we can use the above inequality conclusively, where A_n is the event that an open path exists from 0 to $\partial S(n)$ which is obviously an increasing event.

Suppose A_n occurs. Note that the pivotal edges for A_n will be *uniquely* ordered e_1,e_2.....e_N. This order is the sequence in which these pivotal egdes will be visited in the open path from 0 to $\partial S(n)$ which will be unique.

Also note that in the open path from origin to $\partial S(n)$, either two successive pivotal edges e_i, e_{i+1} will be consecutive or the open component between them is *biconnected* i.e. it has no cut edge. This is because between e_i and e_{i+1} there are no pivotal edges and if the open component between e_i and e_{i+1} has a cut edge, it will definitely be a pivotal edge. To make the above discussion more clear, let us put it more formally. (Also see Figure 1.)

Let $e_i = \langle x_i, y_i \rangle$, where x_i and y_i are the end points of e_i such that in the open path from origin to $\partial S(n)$, x_i is visited before y_i. For every $i \in \{1, 2, ...N\}$ either $y_i = x_{i+1}$ or the open component between e_i and e_{i+1} has no cut edge. The latter is equivalent to the following two statements:

1. The open component between y_i and x_{i+1} is *biconnected*.

2. There are 2 edge disjoint paths between y_i and x_{i+1}.

For each $i \in \{1, 2, ...N\}$, let ρ_i denote $\delta(y_{i-1}, x_i)$, where $y_0 = 0$ and $\delta(u, v)$ is the smallest number of edges required to traverse between u and v.

Let M denote $\max\{k : A_k \text{ occurs}\}$ i.e. M is the radius of the *largest* ball whose boundary contains a vertex having an open path to origin.Note that

$$\mathrm{P}_p(M \geq m) = g_p(m) \quad (2)$$

because the event $M \geq m$ is equivalent to the event that the largest ball, whose boundary contains a vertex having an open path to origin, has radius *atleast* m which in turn is equivalent to saying that there is *atleast* an open path from origin to the boundary of the ball of radius m.

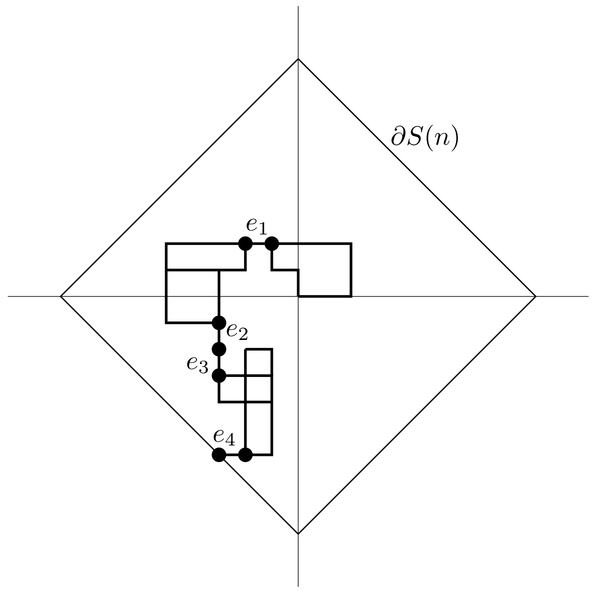

Figure 1: Sequence of critical edges and biconnected components for the event A_n

Lemma 6.2 *Given a non-negative integer $r \leq n-1$. Then for any $p \in (0,1)$:*

$$P_p(\rho_1 \leq r|A_n) \quad \geq \quad P_p(M \leq r).$$

Before proceeding to the proof of this lemma, it is important to understand that the events $(\rho_1 \leq r|A_n)$ and $(M \leq r)$ are related only *mathematically* and with respect to their probability measures. The relationship between their probabilities doesn't imply that there exists some sort of subset/superset relationship between them. After this disclaimer, let's prove the lemma.

Proof of Lemma 6.2: Consider the event: $(\rho_1 > r) \cap A_n$. If $\rho_1 > r$ i.e. $\rho_1 \geq r+1$, then two edge disjoint open paths exist between origin and $\partial S(r+1)$.

Since $r+1 \leq n$, $(\rho_1 > r) \cap A_n$ implies there are edge disjoint open paths from origin to $\partial S(r+1)$ and $\partial S(n)$. This is because of the existence of two edge disjoint open paths from origin to $\partial S(r+1)$. Even if one of these paths goes to $\partial S(n)$, there is still an open path from origin to $\partial S(r+1)$ which is edge disjoint from the former. So we have,

$$\{\rho_1 > r\} \cap A_n \quad \rightarrow \quad A_{r+1} \circ A_n. \tag{3}$$

This is same as saying that

$$\{\rho_1 > r\} \cap A_n \quad \subseteq \quad A_{r+1} \circ A_n.$$

It follows that

$$\mathrm{P}_p(\{\rho_1 > r\} \cap A_n) \quad \leq \quad \mathrm{P}_p(A_{r+1} \circ A_n).$$

By using BK inequality, we get

$$\mathrm{P}_p(\{\rho_1 > r\} \cap A_n) \quad \leq \quad \mathrm{P}_p(A_{r+1}) \cdot \mathrm{P}_p(A_n).$$

Rearranging terms, we obtain

$$\frac{\mathrm{P}_p(\{\rho_1 > r\} \cap A_n)}{\mathrm{P}_p(A_n)} \quad \leq \quad \mathrm{P}_p(A_{r+1}).$$

This is same as saying that

$$\mathrm{P}_p(\{\rho_1 > r\}|A_n) \quad \leq \quad \mathrm{P}_p(A_{r+1}).$$

When complements of the events on both sides are taken, the inequality reverses signs and since $\mathrm{P}_p(A_{r+1})$ is same as $g_p(r+1)$ we get

$$\mathrm{P}_p(\{\rho_1 \leq r\}|A_n) \quad \geq \quad 1 - g_p(r+1).$$

Applying equation (2) we obtain

$$\mathrm{P}_p(\{\rho_1 \leq r\}|A_n) \quad \geq \quad 1 - \mathrm{P}_p(M \geq (r+1)).$$

It follows that

$$\mathrm{P}_p(\{\rho_1 \leq r\}|A_n) \quad \geq \quad \mathrm{P}_p(M \leq r).$$

Note that the converse of equation (3) is not true i.e. $A_{r+1} \circ A_n$ doesn't necessarily imply $\{\rho_1 > r\} \cap A_n$. The counterexample can be seen in Figure 2 which illustrates an outcome of a percolation experiment.

In Figure 2, let the only open edges be the ones that have been highlighted. The event $A_{r+1} \circ A_n$ is occuring because there are two edge disjoint open paths from origin to $\partial S(n)$ and $\partial S(r+1)$. But $\rho_1 = 0$ and therefore $\rho_1 \leq r$. So it is clear that $A_{r+1} \circ A_n$ doesn't imply $\{\rho_1 > r\} \cap A_n$. ∎

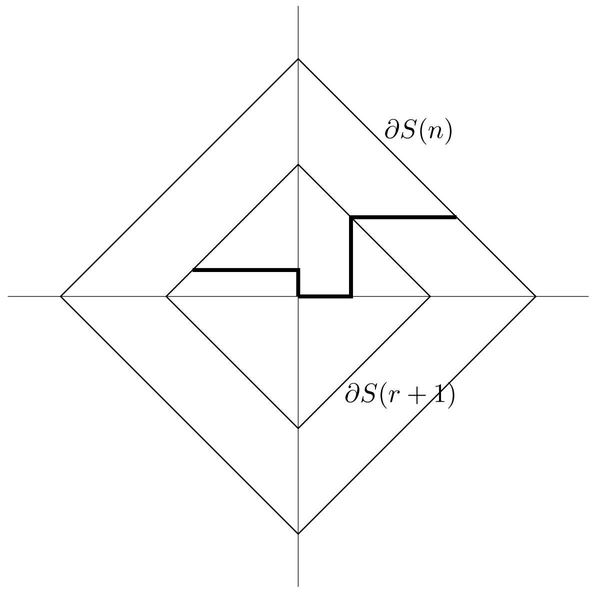

Figure 2: Counter-example to converse of equation (3)

Lemma 6.3 *Given $k > 0$ and non negative integers $r_1, r_2 \ldots r_k$ such that $\sum_{i=1}^{k} r_k \leq n - k$. Then, for $0 < p < 1$,*

$$P_p(\rho_k \leq r_k, \rho_i = r_i, 1 \leq i < k | A_n) \geq P_p(M \leq r_k) \cdot P_p(\rho_i = r_i, 1 \leq i < k | A_n).$$

Proof of Lemma 6.3: Note that Lemma 6.2 was a special case ($k = 1$) of this Lemma. Here we outline the proof for a general k. Let D_e be the set of all vertices reachable from origin along open paths without using e.

Definition 6.4 *Define event B_e for an edge e=$\langle u, v \rangle$ as follows :*

1. *Exactly one of u, v is in D_e, say u*
2. *e is open*
3. *D_e contains no vertex of ∂S_n*
4. *The pivotal edges for $\{0 \leftrightarrow v\}$ are in order $\langle x_1, y_1 \rangle, \langle x_2, y_2 \rangle \ldots \langle x_{k-1}, y_{k-1} \rangle$, where $x_{k-1} = u$ and $y_{k-1} = v$*

5. $\delta(y_{i-1}, x_i) = r_i$ *,where* $1 \leq i < k$ *and* $y_0 = 0$.

Let $B = \cup B_e$ Notice that for a particular outcome w, B_e can occur for only one edge e in the lattice. This follows from the uniqueness of ordering of pivotal edges, as explained in the beginning of the section.

Suppose outcome $w \in A_n \cap B$. Let e be the unique edge such that $w \in B_e$. Construct graph $G = (V', E')$ where $V' = D_e \cup \{v\}$ and $E' = \{(x, y) \mid x \in V', y \in V'\}$. We call v as $y(G)$ and also mark it in the graph. Now, using the concept of marginal distribution,

$$\mathrm{P}_p(A_n \cap B) = \sum_{\varrho} \mathrm{P}_p((A_n \cap B \cap (G = \varrho)) \quad (4)$$

$$= \sum_{\varrho} \mathrm{P}_p(B, G = \varrho).\mathrm{P}_p(A_n | B, G = \varrho) \quad (5)$$

Note that the for a graph $G = \varrho$, the edge e can be different, depending on the percolation outcome w. Therefore, to differentiate the two instances of G, the vertex $y(G)$ has been marked, thus giving independent identities to the two graphs, depending upon the unique edge responsible. Now consider $P(A_n | B, G = \varrho)$.

Claim 6.5 *The event* $\{A_n | B, G = \varrho\}$ *is the same as the event* $\{y(\varrho) \leftrightarrow \partial S_n$ *off* $\varrho.\}$

Let A_n occur given that B occurs and $G = \varrho$. Since, the edge e is a pivotal edge and the event A_n occurs, $y(\varrho)$ is connected to ∂S_n without crossing ϱ. The latter assertion is true, because if a path from $y(\varrho)$ to ∂S_n passes through edge $\langle a, b \rangle$, where $b \in \varrho$, A_n can occur without passing through edge e. This can be done by using the open path $(0, b), (b, \partial S_n)$. Thus e would no longer remain the pivotal edge. Hence $y(\varrho)$ is connected to ∂S_n without crossing ϱ.

In Figure 3, we see an illustration of the argument: if $y(\varrho)$ is connected to $\partial S(n)$ through a path which touches ϱ (the connection is shown with a dotted line line), then e cannot be a pivotal edge, which is a contradiction. Hence Claim 6.5 is proved. ■

From Claim 6.5 and (5), we get

$$\mathrm{P}_p(A_n \cap B) = \sum_{\varrho} \mathrm{P}_p(B, G = \varrho) \cdot \mathrm{P}_p(y(\varrho) \leftrightarrow \partial S_n \text{ off } \varrho) \quad (6)$$

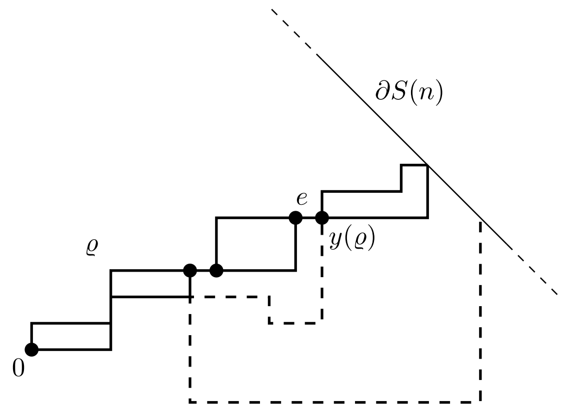

Figure 3: $y(\varrho)$ not connected to ∂S_n off ϱ contradicts the pivotality of e.

Now, similar to (5), the following equation holds :

$$\mathrm{P}_p(\{\rho_k > r_k\} \cap A_n \cap B) = \sum_{\varrho} \mathrm{P}_p(B, G = \varrho) \cdot \mathrm{P}_p(\{\rho_k > r_k\} \cap A_n | B, G = \varrho) \quad (7)$$

Now let event $\{\rho_k > r_k\} \cap A_n$ occur. $\rho_k = \delta(y(\varrho), x_k)$, where x_k could lie on ∂S_n, or be the endpoint of the kth pivotal edge. Also, let $S(a, b)$ denote the set of points at distance $\leq b$ from point a. Since, $\rho_k > r_k$, the event $\{y(\varrho) \leftrightarrow \partial S(y(\varrho), r_k + 1)\}$ occurs. Also, since A_n occurs, the event $\{y \leftrightarrow \partial S_n\}$ occurs.

The pivotality of e ensures that both the above events use edges outside ϱ. Moreover, since there are no pivotal edges between e and $\langle x_k, y_k \rangle$, this ensures that there are two edge disjoint paths from $y(\varrho)$ to x_k. One of them can be used for the event $\{y(\varrho) \leftrightarrow \partial S(y(\varrho), r_k + 1)\}$ and the other for event A_n.

In Figure 4 we see an illustration of this argument. Between the second vertex of the $k - 1$th pivotal edge i.e. $y_{k-1} = y(\varrho)$ and the first vertex of the kth pivotal edge, lie two edge disjoint paths. One can be seen as part of a path that extends to $\partial S(n)$ and the other can be seen to be a path from $y(\varrho)$ to $\partial S(y(\varrho), r_k + 1)$.

From the above analysis,

$$(\{\rho_k > r_k\} \cap A_n | B, G = \varrho) \subseteq (y \leftrightarrow \partial S(y(\varrho), r_k + 1) \text{ off } \varrho) \circ (y(\varrho) \leftrightarrow \partial S_n \text{ off } \varrho)$$

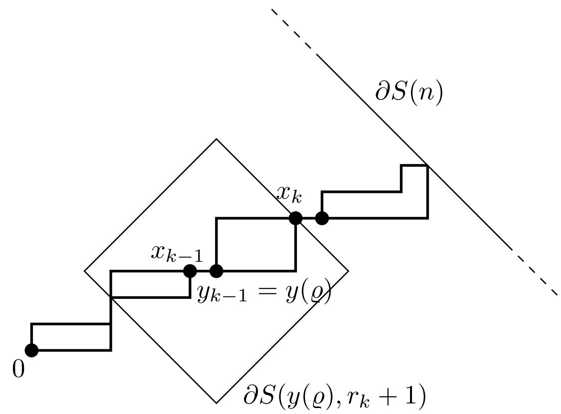

Figure 4: Disjoint Paths to ∂S_n and $\partial S(y(\varrho), r_k + 1)$

Now applying BK Inequality to the RHS of the above equation, we get

$$\mathrm{P}_p(\{\rho_k > r_k\} \cap A_n | B, G = \varrho) \leq \mathrm{P}_p(y \leftrightarrow \partial S(y(\varrho), r_k + 1) \text{ off } \varrho) \cdot \mathrm{P}_p(y(\varrho) \leftrightarrow \partial S_n \text{ off } \varrho)$$

Using Equations (7) and (8) ,we obtain

$$\begin{aligned}\mathrm{P}_p(\{\rho_k > r_k\} \cap A_n \cap B) &\leq \sum_{\varrho} \mathrm{P}_p(B, G = \varrho) \cdot \mathrm{P}_p(y \leftrightarrow \partial S(r_k + 1, y(\varrho)) \text{ off } \varrho) \\ &\quad \cdot \mathrm{P}_p(y(\varrho) \leftrightarrow \partial S_n \text{ off } \varrho) \qquad (8)\end{aligned}$$

Using translation invariance, the $2nd$ term of the RHS above can be brought out common and can be replaced by $\mathrm{P}_p(A_{r_k+1}) = g_p(r_k+1)$. This is because,

$$\mathrm{P}_p(y \leftrightarrow \partial S(r_k + 1, y(\varrho)) \text{ off } \varrho) \leq \mathrm{P}_p(y \leftrightarrow \partial S(r_k + 1, y(\varrho))) \qquad (9)$$

and then we can apply translation invariance. Hence, using this finding and (6), we get

$$\mathrm{P}_p(\{\rho_k > r_k\} \cap A_n \cap B) \leq g_p(r_k + 1)\mathrm{P}_p(A_n \cap B)$$

By some manipulation of the above equation, we get:

$$\mathrm{P}_p(\{\rho_k \leq r_k\} | B \cap A_n) \geq 1 - g_p(r_k + 1) \qquad (10)$$

Multiplying both sides of the equation with $P(B|A_n)$,and using the Definition of B and in particular B_e, we get

$$\mathrm{P}_p(\rho_k \leq r_k, \rho_i = r_i \, for \, 1 \leq i < k | A_n) \geq \mathrm{P}_p(M \leq r_k) \cdot \mathrm{P}_p(\rho_i = r_i \, for \, 1 \leq i < k | A_n)$$

Hence, the Lemma 6.3 is now proved. ■

In our goal to prove Theorem 6.1, we would actually use a variant of the above Theorem i.e

Lemma 6.6 *Given $k > 0$ and non negative integers i and r_k such that $i + r_k \leq n - k$. Then, for $0 < p < 1$,*

$$P_p(\rho_k \leq r_k, \rho_1+\rho_2+\ldots+\rho_{k-1} = i|A_n) \geq P_p(M \leq r_k)\cdot P_p(\rho_1+\rho_2+\ldots+\rho_{k-1} = i|A_n).$$

Proof of Lemma 6.6: The proof for the Lemma above remains exactly the same as for Lemma 6.3. The only change is in Definition of B_e, in which we replace condition 5 with the following condition 5'.

$$\rho_1 + \rho_2 \ldots \rho_{k-1} = i.$$

■

The Corollary below would be directly used in the proof of Theorem 6.1, and would use the Lemma stated above.

Corollary 6.7

$$P_p(\rho_1 + \rho_2 + \rho_3 ... \rho_k \leq n - k|A_n) \geq P_p(M_1 + M_2 \ldots M_k \leq n - k)$$

where $M_1, M_2 \ldots$ is a sequence of independent random variables distributed as M.

Proof of Corollary 6.7: Using the concept of marginal distribution,

$$\mathrm{P}_p(\rho_1 + \rho_2 + \ldots \rho_k \leq n - k|A_n) = \sum_{i=0}^{n-k} \mathrm{P}_p(\rho_1 + \rho_2 \ldots \rho_{k-1} = i, \rho_k \leq n - k - i|A_n)$$

Using the Claim made in Lemma 6.6, we get

$$\begin{aligned} \mathrm{P}_p(\rho_1 + \rho_2 + \ldots \rho_k \leq n - k|A_n) &\geq \sum_{i=0}^{n-k} P(M \leq n - k - i)\mathrm{P}_p(\rho_1 + \rho_2 + \ldots \rho_{k-1} = i|A_n). \\ &\geq \mathrm{P}_p(\rho_1 + \rho_2 \ldots \rho_{k-1} + M_k \leq n - k|A_n) \qquad (11) \end{aligned}$$

We iterate similar to above, replacing ρ_i in each step by M_i, and get the RHS of the Corollary. ■

Having proved the above Corollary, we are now in a position to move forward with our aim to prove Theorem 6.1. We prove another Lemma below in this direction.

Lemma 6.8 *For 0 < p < 1,*

$$E_p(N(A_n)|A_n) \geq \frac{n}{\left(\sum_{i=0}^{n} g_p(i)\right)} - 1$$

Proof of Lemma 6.8: Note that if $\rho_1 + \rho_2 \dots \rho_k \leq n - k$, then $\delta(0, y_k) \leq n - k + k \leq n$. This implies that even by using the first k pivotal edges, we can atmost just reach ∂S_n. Thus, to reach ∂S_n, the number of pivotal edges required $\geq k$. Hence, $N(A_n) \geq k$. Hence,

$$\begin{aligned} \mathrm{P}_p(N(A_n) \geq k|A_n) &\geq \mathrm{P}_p(\rho_1 + \rho_2 + \rho_3 ... \rho_k \leq n - k|A_n) \\ &\geq P(M_1 + M_2 \dots M_k \leq n - k) \end{aligned} \tag{12}$$

The second step above uses the Corollary 6.7, which was proved prior to this Lemma. Now, using the definition of Expectation value,

$$E_p(N(A_n)|A_n) = \sum_{k=1}^{\infty} \mathrm{P}_p(N(A_n \geq k)|A_n) \tag{13}$$

$$\geq \sum_{k=1}^{\infty} P(M_1 + M_2 \dots M_k \leq n - k) \tag{14}$$

The second step follows from Equation (12)

Now, since $P(M \geq r) = g_p(r) \to \theta(p)$ as $r \to \infty$, we work with $M_i' = 1 + \min(M_i, n)$, as it lumps all large values at one place. We would now need to rewrite equation (12) in terms of M_i'. Let the event $\{M_1 + M_2 \dots \leq n - k\}$ occur. Since, $M_i \geq 0, \forall i \geq 0$, thus $M_i \leq n$, $\forall i \in [1, n]$. Hence, $M_i' = 1 + M_i$, $\forall i \in [1, n]$. We thus get that

$$\{M_1 + M_2 \dots \leq n - k\} \subseteq \{M_1' + M_2' \dots M_k' \leq n\} \tag{15}$$

The reverse subset relationship also holds in Equation (15) , and the argument follows similar lines. Hence, the following equality holds true :

$$\mathrm{P}_p(M_1 + M_2 \dots \leq n - k) = \mathrm{P}_p(M_1' + M_2' \dots M_k' \leq n) \tag{16}$$

Rewriting (12) using (16) ,we get

$$E_p(N(A_n)|A_n) \geq \sum_{k=1}^{\infty} P(M_1' + M_2' \dots M_k' \leq n) \tag{17}$$

Now define, $K = \min\{k : M'_1 + M'_2 \dots M'_k > n\}$. From the definition of K,

$$\{M'_1 + M'_2 \dots M'_k \leq n\} \leftrightarrow \{K \geq k+1\} \tag{18}$$

Using (17) and (18),

$$\begin{aligned} E_p(N(A_n)|A_n) &\geq \sum_{k=1}^{\infty} P(K \geq (k+1)) \\ &= E(K) - 1 \end{aligned} \tag{19}$$

Now, note that K is a stopping time for the sequence M'_1,$M'_2 \dots M'_k$.
Hence if $S_k = M'_1 + M'_2 \dots M'_k$, then using Wald's Equation (as described in Theorem 6.13)

$$E(S_K) = E(K)E(M'_1).$$

Since $S_K > n$ (by Definition of K), we get

$$E(K) > \frac{n}{(E(M'_1)} \tag{20}$$

To proceed further, we need to calculate $E(M'_i)$.

$$\begin{aligned} E(M'_1) &= 1 + E(\min(M_1, n)) \\ &= 1 + \sum_{i=1}^{n} i.P(M_1 = i) + \sum_{i=n+1}^{\infty} n.P(M_1 = i) \\ &= 1 + \sum_{i=1}^{n} P(M_1 \geq i) \\ &= \sum_{i=0}^{n} P(M_1 \geq i) \\ &= \sum_{i=0}^{n} g_p(i) \end{aligned} \tag{21}$$

The $2nd$ last step follows from the fact that $P(M_1 \geq 0) = 1$. Using Equations (19), (20) and (21), we get

$$E_p(N(A_n)|A_n) \geq \frac{n}{\left(\sum_{i=0}^{n} g_p(i)\right)} - 1$$

The Lemma 6.8 is thus proved.

Note that the inequality in the Claim above holds for all values $0 < p < 1$. However, for $p > p_c$, due to the presence of an infinite giant component, $\sum g_p(i)$ will be quite large and the lower bound will be very weak. Thus the inequality is more useful for the case $p < p_c$. ■

The bound above on $E(N(A_n))$ helps us bound the right hand side of the Equation (1). We thus get the following inequality :

$$g_\alpha(n) \leq g_\beta(n) \exp\left(-\int_\alpha^\beta \left(\frac{n}{\sum_{i=0}^n g_p(i)} - 1\right) dp\right) \tag{22}$$

Since, $g_p(i) \leq g_\beta(n) \forall p \leq \beta$, the integral on the RHS can be upperbounded to yield :

$$g_\alpha(n) \leq g_\beta(n) \exp\left(-(\beta - \alpha)\left(\frac{n}{\sum_{i=0}^n g_\beta(i)} - 1\right)\right) \tag{23}$$

The way, the proof will now proceed is that we would introduce a very weak bound for $g_p(i)$. Using that bound and the equation above, we would have a better and tigher bound on $g_p(i)$.

Lemma 6.9 *For $p < p_c$, there exists a $\delta(p)$ such that*

$$g_p(n) \leq \frac{\delta(p)}{\sqrt{n}} \tag{24}$$

We would not be proving the above Lemma and instead use it to prove our next Corollary.

Corollary 6.10 *There is a finite quanity $c(\alpha)$ such that for all $\alpha < p_c$*

$$\sum_{i=0}^{\infty} g_\alpha(i) < c(\alpha) \tag{25}$$

Proof of Corollary 6.10: From Lemma 6.9,

$$\begin{aligned}
\sum_{i=0}^{n} g_p(i) &\leq \sum_{i=0}^{n} \frac{\delta(p)}{\sqrt{i}} \\
&\leq \delta(p) \int_0^n \frac{1}{\sqrt{x}}\, dn \\
&= \triangle(p)\sqrt{n} \qquad (26)
\end{aligned}$$

The inequality in the second step derives from the fact that, the integral corresponds to the area under the curve $f(x) = \frac{1}{\sqrt{n}}$ from 0 to n, which is obviously greater than sum of some individual values of $f(x)$ from 0 to n.

Now, given $\alpha < p_c$, take some β such that $\alpha < \beta < p_c$. Using Equation (26),

$$\sum_{i=0}^{n} g_\beta(i) \leq \triangle(\beta)\sqrt{n} \qquad (27)$$

Plugging the above inequality into Equation (23), we get

$$\begin{aligned}
g_\alpha(n) &\leq g_\beta(n) \exp\left(1 - \frac{\beta - \alpha}{\triangle(\beta)}\sqrt{n}\right) \\
&\leq \exp\left(1 - \frac{\beta - \alpha}{\triangle(\beta)}\sqrt{n}\right)
\end{aligned}$$

Taking a summation on both sides, we get

$$\sum_{i=0}^{\infty} g_\alpha(i) \leq \sum_{i=0}^{\infty} \frac{e}{\exp\left(\frac{\beta-\alpha}{\triangle(\beta)}\sqrt{i}\right)} \qquad (28)$$

The series on the RHS will converge to a finite value $c(\alpha)$, and hence we have an upperbound for $\sum_{i=0}^{\infty} g_\alpha(i)\ \forall \alpha < p_c$.

The above Corollary is hence proved. ∎

We are now in a position to finally prove Theorem 6.1. Using Equation (23), for an $\alpha < p_c$, we can pick $\beta > \alpha$ and $< p_c$, such that

$$\begin{aligned} g_\alpha(n) &\leq g_\beta(n)\exp\left(-(\beta-\alpha)\left(\frac{n}{\sum_{i=0}^{n} g_\beta(i)} - 1\right)\right) \\ &\leq g_\beta(n)\exp\left(1 - \frac{\beta-\alpha}{c(\beta)}(n)\right) \\ &\leq \exp\left(1 - \frac{\beta-\alpha}{c(\beta)}(n)\right) \end{aligned}$$

The second step above uses Corollary 6.10 and the last step follows from the fact that $g_\beta \leq 1$. Since, $\alpha < \beta < p_c$, we can upperbound the RHS of the equation above by $e^{-n\psi(\alpha)}$. Notice that a similar logic was also used after Equation (28).
Hence, we have finally proved the Theorem 6.1 that for $0 < \alpha < p_c$,

$$g_\alpha(n) \leq e^{-n\psi(\alpha)}$$

■

6.2 The expectation of the size of the open cluster is finite

We are now in a position to show that the expectation of the open cluster is finite in the subcritical phase.

Theorem 6.11 *If* $p < p_c$ *then*

$$\chi(p) < \infty.$$

Proof. First, let us note that for a d-dimensional mesh there is a constant c such that

$$|S(n)| \leq cn^d.$$

Since, the probability $\mathrm{P}_p(M < \infty) = 1$ when $p < p_c$, we can say that

$$\chi(p) = \mathrm{E}_p(|C|) = \sum_{n=1}^{\infty} \mathrm{E}_p(|C| \mid M = n)\mathrm{P}_p(M = n).$$

But if $M = n$ i.e. the largest diamond to which the origin is connected is $S(n)$ then $|C|$ is upper bounded by $|S(n)|$. Hence we can say

$$\chi(p) \leq \sum_{n=1}^{\infty} cn^d \mathrm{P}_p(M = n).$$

Now, using Theorem 6.1 we get

$$\chi(p) \leq \sum_{n=1}^{\infty} cn^d . e^{-n\Psi(p)}.$$

Hence proving that $\chi(p)$ is finite. ∎

6.3 Appendix: Wald's Equation

Definition 6.12 *Let X_1,X_2,.... be a sequence of random variables. The non-negative integer valued random variable N is said to be a* stopping time *of the sequence $\{X_n\}$ if $\forall n \in \mathbb{N}$ the event $\{N = n\}$ is independent of $X_i, i \geq n+1$.*

For example, if $\{X_n\}$ is a sequence of i.i.d. random variables such that

$$\forall i : P(X_i = 1) = p \text{ and } P(X_i = 0) = 1 - p$$

then, $N_1 = \min\{n : X_1 + X_2 + ...X_n = 5\}$ is a stopping time for $\{X_n\}$.

$$N_2 = \begin{cases} 3 & \text{if } X_1 = 0, \\ 1 & \text{if } X_1 = 1. \end{cases}$$

is also a stopping time for $\{X_n\}$. But

$$N_3 = \begin{cases} 3 & \text{if } X_4 = 0, \\ 1 & \text{if } X_4 = 1. \end{cases}$$

is not a stopping time for $\{X_n\}$.

The expectations of the sum of a random prefix of the sequences of i.i.d. random variables have a good property, known as Wald's equation.

Theorem 6.13 *If $X_1, X_2, \ldots$ are i.i.d random variables distributed as X, with finite mean ($E(X) < \infty$) and if N is a stopping time of this sequence such that ($E(N) < \infty$) then*

$$E\left(\sum_{i=1}^{N} X_i\right) = E(N)E(X).$$

Proof. Define a sequence of indicators

$$I_i = \begin{cases} 1 & \text{if } i \le N, \\ 0 & \text{if } i > N. \end{cases}$$

For each subscript i, we have a different random variable I_i dependent on random variable N. We Claim that X_i are I_i are independent of each other. To see this, rewrite I_i as

$$I_i = \begin{cases} 0 & \text{if } N \le i-1, \\ 1 & \text{if } N > i-1. \end{cases}$$

From the first condition, the event $I_i = 0$ depends on the event $N \le i-1$, which is independent of $X_i, X_{i+1} \dots$, since N is a stopping time for $\{X_n\}$. Hence, I_i and X_i are independent.

We now use these indicators to say that

$$\sum_{i=1}^{N} X_i = \sum_{i=1}^{\infty} X_i I_i.$$

Taking expectations on both sides,

$$\begin{aligned} \mathrm{E}\left(\sum_{i=1}^{N} X_i\right) &= \mathrm{E}\left(\sum_{i=1}^{\infty} X_i I_i\right) && (29) \\ &= \sum_{i=1}^{\infty} \mathrm{E}(X_i I_i) && (30) \\ &= \sum_{i=1}^{\infty} \mathrm{E}(X_i)\mathrm{E}(I_i) && (31) \\ &= \mathrm{E}(X) \sum_{i=1}^{\infty} \mathrm{E}(I_i) && (32) \end{aligned}$$

The second step can be proved although the summation is infinite, but we omit the proof here. The third step follows from the fact that X_i and I_i are independent of each other

Now, since $\mathrm{E}(I_i) = \mathrm{P}(N \ge i)$ we can say that

$$\mathrm{E}\left(\sum_{i=1}^{N} I_i\right) = \sum_{i=1}^{N} \mathrm{P}(N \ge i) = E(N).$$

putting this back in (32) gives us the result. ■