Lecture 5: Russo's Formula

10th September 2007

5.1 Introduction and Definitions

In this lecture, we derive Russo's Formula, which gives the rate of change of the likelihood of an event (this should be an *increasing* event) happening with change in the percolation probability p (the probability of an edge being open). The statement of the differential form of Russo's formula is:

$$\frac{dP_p(A)}{dp} = E_p(N(A)) \tag{1}$$

Here P_p is the probability measure over the event space given that the percolation likelihood is p; A is an increasing event in the space; E_p denotes the expectation function for the given p; and $N(A)$ is the number of "pivotal" edges (see Section 5.3.2) for the event A.

In the following sections, we lead up to the derivation of this formula, starting with a motivating example.

5.2 An Example event

Let $\partial B(n)$ denote the boundary of the box of side length $2n$ centred at the origin. Let A be the event that there is an open path from the origin to this boundary; we denote this event as $(0 \leftrightarrow \partial B(n))$. Figure 1 shows a situation where the event A has not happened, but is "close" to happening; by flipping one or two of the closed (dashed) edges numbered 1-5, A can be caused to happen. Now, suppose p is the percolation probability, i.e., the probability with which an edge was taken to be open when the random sample represented here was picked. Then, as per the coupling argument introduced in an earlier lecture, we can think of the experiment that generated this outcome as having chosen a random value uniformly in the range $0 - 1$ for each edge; and having set to open those edges for which this value was p or less.

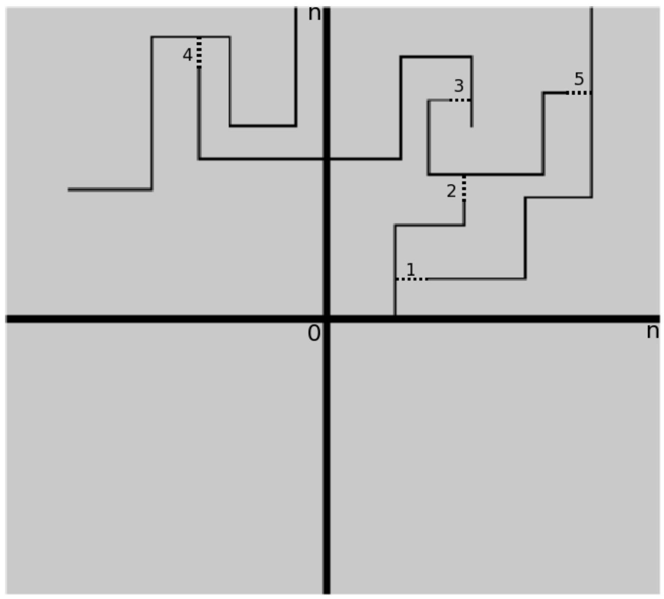

Figure 1: The event $(0 \leftrightarrow \partial B(n))$ has "nearly happened". Open edges are shown in black, and some closed ones are shown dashed.

Consider the outcome shown in Figure 2. Here $X(e_i)$ is the random variable corresponding to labelled edge i from Figure 1. Clearly, all the values must be greater than p, as all these edges are closed. However, now suppose we were to gradually increase the value of p, whilst keeping the experimental outcome the same. So p becomes $p + \delta$, for some $\delta > 0$. When $\delta = d_1$, edge 5 becomes open. The event A has still not happened though. Further increase δ to make it $d_1 + d_2$; now edge 4 is also open, but A is still false. However, if we increase δ to $d_1 + d_2 + d_3$, then edge 2 also becomes open; and now, there is an open path from the origin to $\partial B(n)$ in Figure 1, passing through edges 2 and 5.

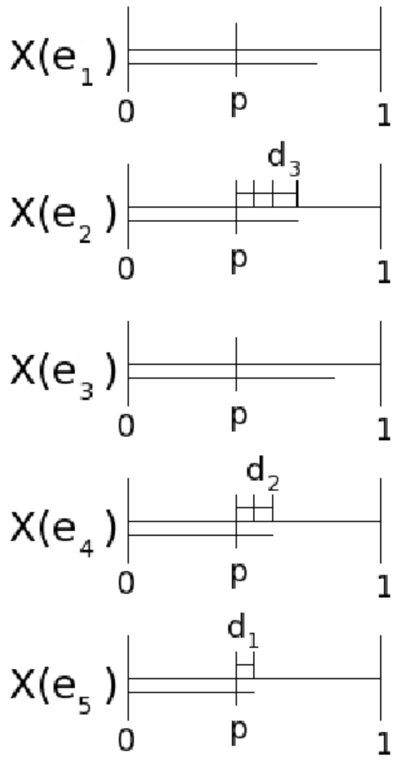

Figure 2: The results of the "experiment" in choosing a random value between 0 and 1 for the 5 dashed edges shown in Figure 1. By increasing the value of p sufficiently, these edges can be caused to become open for the same experiment.

5.3 General Formulation

5.3.1 Notation

For a given percolation probability p, we define an *outcome* as a mapping from the edge set to 1 or 0: $\eta_p : \mathbb{E}^d \to \{1, 0\}$. It is defined as:

$$\eta_p(e) = \begin{cases} 1 & \text{if } X(e) \leq p \\ 0 & \text{otherwise.} \end{cases}$$

5.3.2 Pivotal Sets

We would like to look at how the likelihood of an event happening varies as p increases. We have seen that by adding an increment to p, more edges can open up for a given experimental outcome. The key question is, which of these edges can affect whether a given event happens or does not happen? In order to capture this notion, we define the concept of *pivotal sets* of edges.

Definition 5.1 *A* pivotal set *of edges for a given outcome and a given event A is a set $P \subset \mathbb{E}^d$, such that if all the edges in P are made open whilst keeping the state of all other edges unchanged, A happens. The set P should be minimal, i.e., no proper subset of P should have the same property.*

We will now use this definition of a pivotal set to formulate an expression for the change in the likelihood of an event with change in the percolation probability p.

5.3.3 Derivation of Russo's Formula

Viewing an event A as the set of outcomes that cause it to happen, we can say: $P_{p+\delta}(A) = P(\eta_{p+\delta} \in A)$ (here, P_p denotes the probability measure obtained by setting the percolation likelihood to p). This can be further broken down as:

$$\begin{aligned}
P_{p+\delta}(A) &= P(((\eta_{p+\delta} \in A) \cap (\eta_p \notin A)) \\
&\quad \cup((\eta_{p+\delta} \in A) \cap (\eta_p \in A))) \\
&= P(((\eta_{p+\delta} \in A) \cap (\eta_p \notin A)) \cup (\eta_p \in A)) \qquad (A \text{ is increasing}) \\
&= P((\eta_{p+\delta} \in A) \cap (\eta_p \notin A)) + P_p(A) \qquad (\text{disjoint events})
\end{aligned}$$

Hence we get

$$
\begin{aligned}
P_{p+\delta}(A) - P_p(A) &= P((\eta_{p+\delta} \in A) \cap (\eta_p \notin A)) \\
&= P(\eta_p \notin A).P(\eta_{p+\delta} \in A \mid \eta_p \notin A) \\
&= \sum_{e \in \mathbb{E}^d} P_p((\{e\} \text{ is pivotal for } A) \cap (p \leq X(e) \leq p+\delta)) \\
&\quad + \sum_{e_1, e_2 \in \mathbb{E}^d} P_p((\{e_1, e_2\} \text{ is pivotal for } A) \cap (p \leq X(e_1) \leq p+\delta) \\
&\quad \cap (p \leq X(e_2) \leq p+\delta)) \\
&\quad + \ldots
\end{aligned}
$$

Here we note that for a given outcome, whether a set of edges is pivotal or not *does not* depend on the state (open/closed) of those edges, but only on the state of the remaining edges. So, for instance, whether the set $\{e\}$ is pivotal or not is independent of the value of $X(e)$. Thus, we can re-write the above expression as follows:

$$P_{p+\delta}(A) - P_p(A) = \sum_{e \in \mathbb{E}^d} P_p(\{e\} \text{ is pivotal for } A).\delta + \delta^2(\ldots)$$

The higher-order terms on the right all have an exponent of 2 or more for δ. Now, we will divide both sides by δ and take the limit as δ tends to 0; so all these terms will vanish:

$$
\begin{aligned}
\lim_{\delta \to 0} \frac{P_{p+\delta}(A) - P_p(A)}{\delta} &= \sum_{e \in \mathbb{E}^d} P_p(\{e\} \text{ is pivotal for } A) \\
\frac{dP_p(A)}{dp} &= \sum_{e \in \mathbb{E}^d} P_p(\{e\} \text{ is pivotal for } A)
\end{aligned}
$$

The last step is analogous to the standard notion of a dervative; we skip the rigorous proof of its correctness here. If we take $N(A)$ to denote the number of individual pivotal edges for the event A, then we get the standard form of Russo's formula (1):

$$\frac{dP_p(A)}{dp} = E_p(N(A))$$

5.3.4 Integral form of Russo's Formula

An integral form for (1) can be derived as follows:

$$\begin{aligned}
\frac{dP_p(A)}{dp} &= \sum_{e\in\mathbb{E}^d} P_p(\{e\} \text{ is pivotal for } A) \\
&= \frac{1}{p}\sum_{e\in\mathbb{E}^d} P_p((\{e\} \text{ is pivotal for } A) \cap (e \text{ is open})) \\
&= \frac{1}{p}\sum_{e\in\mathbb{E}^d} P_p(\{e\} \text{ is pivotal for } A \mid A).P_p(A) \\
&= \frac{1}{p}E_p(N(A) \mid A).P_p(A)
\end{aligned}$$

Thus we get:

$$P_{p_2}(A) = P_{p_1}(A)\exp\left(\int_{p_1}^{p_2} \frac{1}{p}E_p(N(A) \mid A)dp\right) \tag{2}$$

5.4 Example Usage of Russo's Formula

Suppose we take a one-dimensional graph (i.e., a line); let A denote the event that there is an open path from 0 to n ($0 \leftrightarrow n$). Since all edges between 0 and n must be open for A to be true, the value of $N(A)$ given that A has happened is always n. Using (2), we get:

$$\begin{aligned}
P_{p_2}(A) &= P_{p_1}(A)\exp\left(n\int_{p_1}^{p_2}\frac{dp}{p}\right) \\
&= P_{p_1}(A)\exp\left(n\log\frac{p_2}{p_1}\right) \\
\frac{P_{p_2}(A)}{P_{p_1}(A)} &= \left(\frac{p_2}{p_1}\right)^n
\end{aligned}$$

This gives an indication of the growth rate of the likelihood of A with increase in the percolation probability. In general, for any event dependent on n edges, given that the event has happened, there can be at most n pivotal edges. So, we can generalize the above result to say that for $0 < p_1 < p_2 < 1$

$$\frac{P_{p_2}(A)}{P_{p_1}(A)} \leq \left(\frac{p_2}{p_1}\right)^n. \tag{3}$$

where A is any increasing event dependent on a fixed number of edges, and this number is n.