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Lecture 4: BK inequality

27th August and 6th September, 2007

4.1 Preliminaries

The FKG inequality allows us to lower bound the probability of the intersection of two increasing events. However, getting a general upper bound is difficult. So we try to upper bound a different kind of event which is contained in the intersection. Let us try and motivate this through an example.

In this lecture unless otherwise stated, all paths mentioned are open paths.

Given $x, y, u, v \in \mathbb{Z}^2$, we define the following events:

- A : There exists a path between the nodes x and y
- B : There exists a path between the nodes u and v
- $A \cap B$: There exists a path between the nodes x and y AND a path between u and v

Note that for an outcome that satisfies the event $A \cap B$, the two paths in question might overlap on some edges (we focus only on edge overlaps not on vertext overlaps). However, we can think of a subset of these outcomes where this is not the case. We use the notation $A \circ B$ to denote this i.e. we define $A \circ B$ as the event where the edges used by the $x - y$ path are disjoint from those used by the $u - v$ path.

It is easy to see that

$$A \circ B \subseteq A \cap B,$$

which implies that

$$P(A \circ B) \leq P(A \cap B).$$

Let us try to formalize this notation. Given an outcome $\omega \in \Omega$, recall that $K(\omega) = \{e : \omega(e) = 1\}$. Then, we can say that for increasing events A and B, $\omega \in A \circ B$, if there exists ω_1, ω_2 such that,

- $\omega_1 \in A, \omega_2 \in B$
- $K(\omega_1) \cap K(\omega_2) = \emptyset$
- $K(\omega_1) \cup K(\omega_2) \subseteq K(\omega)$

i.e. the set of open edges of ω can be partitioned into two parts such that the two events involved can be satisfied by assigning one partition each to them and giving them use only of the edges in that partition.

Why have we constrained this definition to increasing events? Consider the following non-increasing events:

- E : There exists an even number of open edges.
- O : There exists an odd number of open edges.

Here, $E \cap O = \emptyset$, but $E \circ O \neq \emptyset$. Take $\omega, \omega_1, \omega_2$ such that $|K(\omega)| = 17$, $|K(\omega_1)| = 10$ and $|K(\omega_2)| = 7$.

4.2 BK inequality

We will mainly use the BK inequality in the bond percolation setting, but let us state it in slightly more general way.

Given a positive integer m, let $\Gamma = \prod_{i=1}^{m}\{0,1\}$ and $\mathcal{F}$ be the set of all subsets of Γ. Let P be a product measure on $(\Gamma, \mathcal{F})$ with density $p(i)$ on the i-th coordinate of vectors on Γ i.e. defining $\mu_i(1) = p(i)$ and $\mu_i(0) = 1 - p(i)$,

$$P = \prod_{i=1}^{m} \mu_i.$$

The following result, proved by van den Berg and Kesten in 1985, is referred to as the BK inequality

Theorem 4.1 (BK Inequality) *For two increasing events A and B,*

$$P(A \circ B) \leq P(A) \cdot P(B).$$

Proof. We begin by taking the probability space $(\Gamma, \mathcal{F}, P)$ and producing two identical copies of it: $\mathcal{S}_1 = (\Gamma_1, \mathcal{F}_1, P_1)$ and $\mathcal{S}_2 = (\Gamma_2, \mathcal{F}_2, P_2)$. The product space of these two spaces, $\mathcal{S}$ is defined as

$$\mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2 = (\Gamma_1 \times \Gamma_2, \mathcal{F}_1 \times \mathcal{F}_2, P_{12}),$$

where $P_{12} = P_1 \cdot P_2$ i.e. if $A_1 \in \mathcal{F}_1$, and $A_2 \in \mathcal{F}_2$, then $A_{12} = A_1 \times A_2 \in \mathcal{F}_1 \times \mathcal{F}_2$, and $P(A_{12}) = P_1(A_1)P_2(A_2)$.

We will write $x \times y$ for a typical point in $\Gamma_1 \times \Gamma_2$ with $x = (x_1, \dots, x_m)$ being a point in Γ_1 and $y = (y_1, \dots, y_m) \in \Gamma_2$.

Based on the two increasing events A, B in the original space $(\Gamma, \mathcal{F}, P)$, we define the following events in the product space: A' is the set of all points $x \times y \in \Gamma_1 \times \Gamma_2$ such that $x \in A$. Note that $A' = A \times \Gamma_2$.

Also, B'_k is the set of all points $x \times y$ such that the composite vector $(y_1, \dots, y_k, x_{k+1}, \dots, x_m) \in B$, $0 \leq k \leq m$.

Let us take an example. Consider the event $B = \{0110, 0111, 1111, 1110\}$. In words we can say that B is the event that the second and third component of the outcome vector are both 1. Then, $(0000, 1111) \in B'_3$ because the composite vector for the subscript 3 is 1110 which has the required condition. On the other hand $(0000, 1111)$ is not in B'_1 because the composite vector in that case is 1000.

Note that, $B'_0 = B \times \Gamma_2$ i.e. the set of point $x \times y \in \Gamma_1 \times \Gamma_2$ such that $x \in B$ in the original space Γ. Hence we can now see that

$$P(A \circ B) = P_{12}(A' \circ B'_0) \tag{1}$$

Similarly, $B'_m = \Gamma_1 \times B$. And this implies that $A' \circ B'_m = A' \cap B'_m$ since the two events A' and B'_m depend on disjoint sets of points. Using this fact and the fact that P_{12} is a product measure we get,

$$\begin{aligned} P_{12}(A' \circ B'_m) &= P_{12}(A' \cap B'_m) && (2) \\ &= P_{12}((A \times \Gamma_2) \cap (\Gamma_1 \times B)) && (3) \\ &= P(A)P(B) && (4) \end{aligned}$$

In order to complete the proof we relate $P_{12}(A' \circ B'_0)$ and $P_{12}(A' \circ B'_m)$ in the following way:

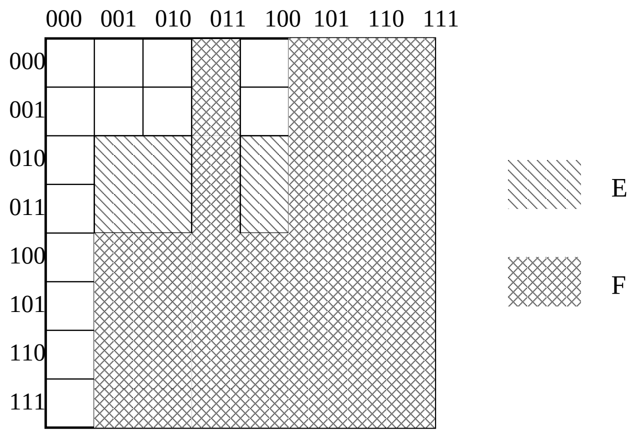

Figure 1: A pictorial depiction of $\Gamma_1 \times \Gamma_2$. A is the event that there is at least one 1. B is the same as A. $E = (A' \circ B_2) \setminus (A' \circ B_1)$, $F = (A' \circ B_1') \cap (A' \circ B_2') = A' \circ B_1'$.

Claim 4.2

$$P_{12}(A' \circ B_{k-1}') \leq P_{12}(A' \circ B_k'), \ \textit{for } 1 \leq k \leq m.$$

If this claim holds, it follows from (1) and (2-4) that

$$P(A \circ B) = P_{12}(A' \circ B_0') \leq P_{12}(A' \circ B_m') = P(A)P(B).$$

And this completes the proof of the BK inequality. ■

We have omitted the proof of Claim 4.2 here. But let us try and understand the intuition behind it. Let us take $m = 3$ i.e. Γ is a sample space with 8 possible outcomes, each a vector with 3 components. Define the event A as follows : At least one of the components is 1. We define the event B to be identical to A.

In Figure 1 we pictorially depict the outcomes corresponding to $A' \circ B_1$ and $A' \circ B_2$. The reader can verify that we find that $A' \circ B_1$ is properly contained in $A' \circ B_2$. We do not claim that this is always true, but it gives us an idea of the direction the proof would take.

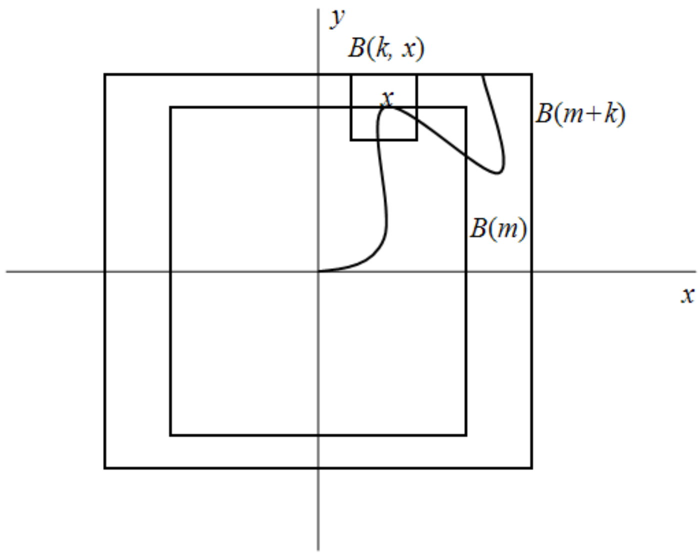

Figure 2: Path crossing $B(m)$ to reach $B(m+k)$

4.3 Applications of BK inequality

4.3.1 Example 1

Suppose X is a random variable and $P[X > n] = k(\alpha)/n^\alpha$.
Then, $\lim_{n\to\infty} P[X > n] = 0$. If $\alpha \le 1, E[X] = \sum_{n=1}^{\infty} P[X > n] \to \infty$ as $n \to \infty$.

Let, $\chi(p) = E_p(|C|)$. Denote by $\partial B(n)$, the boundary of the box $B(n)$. Let $x \leftrightarrow y$ denote that there is an open path between x and y. Then, we have the following theorem.

Theorem 4.3 *If $\chi(p) < \infty$, there exists $\sigma(p) > 0$ such that,*

$$P_p[0 \leftrightarrow \partial B(n)] \le e^{-n\sigma(p)}.$$

Proof. Let the random variable N_n be the number of nodes of $\partial B(n)$ to which the origin is connected. For $x \in \partial B(n)$, define $\tau_p(0, x) = P_p[0 \leftrightarrow x]$.

Then, $E_p[N_n] = \sum_{x\in\partial B(n)} \tau_p(0, x)$. It follows that,

$$0 \leftrightarrow \partial B(m + k) \quad \subseteq \bigcup_{x\in\partial B(m)} (0 \leftrightarrow x) \circ (x \leftrightarrow \partial B(k, x))$$

In Figure 2, $B(k,x)$ is the box centered at $x \in \partial B(m)$ of side length $2k$. The above relationship holds because a path from the origin to a point on the boundary of $B(m+k)$ must intersect the boundary of $B(m)$ at some point x and from x there will be another path to a point on the boundary of $B(k,x)$. Note that these paths are *edge-disjoint*. But not all paths of the latter type are of the former type (e.g. when the path ends inside $B(m+k)$). Hence, the former is a subset of the latter.

From this we get,

$$\begin{aligned}
P_p[0 \leftrightarrow \partial B(m+k)] &\leq \sum_{x\in\partial B(m)} P_p[(0 \leftrightarrow x) \circ (x \leftrightarrow \partial B(k,x))] \\
&\leq \sum_{x\in\partial B(m)} P_p[0 \leftrightarrow x] \cdot P_p[x \leftrightarrow \partial B(k,x)] \\
&= \sum_{x\in\partial B(m)} \tau_p(0,x) \cdot P_p[0 \leftrightarrow \partial B(k)] \\
&= P_p[0 \leftrightarrow \partial B(k)] \sum_{x\in\partial B(m)} \tau_p(0,x) \\
&= E_p[N_m] \cdot P_p[0 \leftrightarrow \partial B(k)].
\end{aligned}$$

The inequality on the second line follows from *BK inequality*, and the equality on the third line follows from *translation invariance*.

Now we are going to establish the relation between $E_p[N_n]$ and $\chi(p)$.

$$\begin{aligned}
\sum_{n=1}^{\infty} E_p[N_n] &= \sum_{n=1}^{\infty} \sum_{x\in\partial B(n)} \tau_p(0,x) \\
&= \sum_{n=1}^{\infty} \sum_{x\in\partial B(n)} E_p[C_x] \\
&= E_p(|C|) \\
&= \chi(p).
\end{aligned}$$

Since, $\sum_{n=1}^{\infty} E_p[N_n] < \infty$, $\lim_{n\to\infty} E_p[N_n] = 0$. Choose an m^* such that, $\eta = E_p[N_{m^*}] < 1$.

Suppose, $n = rm^* + s$, where $r \geq 0,\ 0 \leq s < m^*$.

$$\begin{aligned}
P_p[0 \leftrightarrow \partial B(n)] &\leq P_p[0 \leftrightarrow \partial B(rm^*)] \\
&\leq \eta P_p[0 \leftrightarrow \partial B((r-1)m^*)]
\end{aligned}$$

$$
\begin{aligned}
&\leq \;\; \dots\dots\dots\dots\dots\dots\dots \\
&\leq \;\; \eta^r P_p[0 \leftrightarrow \partial B(0)] \\
&\leq \;\; \eta^{\left(1+\frac{n}{m^*}\right)} \\
&= \;\; e^{-n\sigma(p)}.
\end{aligned}
$$

Here, $\sigma(p) = f(m^*)$ is only a function of p independent of n.

4.3.2 Example 2

Consider the network G depicted in Figure 3. There are n nodes from s to t. Each node is connected to its two neighbours by $\log n$ paralell edges.

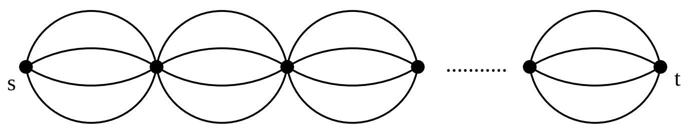

Figure 3: A multi-edged network

Now suppose each edge is removed with probability 1/2, to give a network G'.

Problem 1 *Prove that the min cut between s and t in G' is at most* $\log n/2$ *with probability* $1 - \Theta(\frac{1}{n^\epsilon})$ *for some non-negative ϵ. What is the value of ϵ?*

Solution. We are assuming that all logarithms are in base 2. If the base is different, some coefficients may change.

Lets define the following events.

- A : There exists at least one $s - t$ path.
- A_k: There exist at least k ***edge-disjoint*** $s - t$ paths.

Then, $A_k = A \circ ... \circ A$ (k times). Since A is an increasing event, by BK inequality,

$$P(A_k) \leq P(A)^k$$

Let the vertices of the graph be numbered as $s = v_1, v_2, ..., v_{n-1}, v_n = t$ (from the left). There exists an $s-t$ path if and only if there is at least one edge between each of the nodes v_i and v_{i+1} for $1 \leq i \leq n-1$. Let B_i denote this event. Then,

$$B_i = 1 - \frac{1}{2^{\log n}} = 1 - \frac{1}{n},$$

as $\frac{1}{2^{\log n}}$ is the probability that all the $\log n$ edges will be removed.
Now,

$$A = \bigcap_{i=1}^{n-1} B_i$$

and the B_i's are independent of each other. Therefore,

$$P(A) = \prod_{i=1}^{n-1} P(B_i) = \prod_{i=1}^{n-1} \left(1 - \frac{1}{n}\right) = \left(1 - \frac{1}{n}\right)^{n-1}.$$

We know that for $n \geq 2$,

$$P(A) = \left(1 - \frac{1}{n}\right)^{n-1} \leq \frac{1}{2}.$$

Hence,

$$\begin{aligned} P(A_k) &\leq P(A)^k \\ &\leq 2^{-k}. \end{aligned}$$

Setting $k = \log n/2$ we get,

$$\begin{aligned} P(A_{\log n/2}) &\leq (2^{\log n})^{-1/2} \\ &= \frac{1}{n^{1/2}}. \end{aligned}$$

By Menger's Theorem,

Size of a minimum $s-t$ cut = Maximum number of edge-disjoint $s-t$ paths

From the above discussion, the probability that the number of edge-disjoint $s-t$ paths is at most $\log n/2$ is $1 - \Theta\left(\frac{1}{n^{1/2}}\right)$. Hence, the min-cut between s and t in G' is at most $\log n/2$ with probability $1 - \Theta\left(\frac{1}{n^{1/2}}\right)$, implying $\epsilon = \frac{1}{2}$.